

# Space-Efficient Manifest Contracts

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October 13, 2014

## Abstract

The standard algorithm for higher-order contract checking can lead to unbounded space consumption and can destroy tail recursion, altering a program’s asymptotic space complexity. While space efficiency for gradual types—contracts mediating untyped and typed code—is well studied, sound space efficiency for manifest contracts—contracts that check stronger properties than simple types, e.g., “is a natural” instead of “is an integer”—remains an open problem.

We show how to achieve sound space efficiency for manifest contracts with strong predicate contracts. The essential trick is breaking the contract checking down into *coercions*: structured, blame-annotated lists of checks. By carefully preventing duplicate coercions from appearing, we can restore space efficiency while keeping the same observable behavior.

Along the way, we define a framework for space efficiency, traversing the design space with three different space-efficient manifest calculi. We examine the diverse correctness criteria for contract semantics; we conclude with a coercion-based language whose contracts enjoy (galactically) bounded, *sound* space consumption—they are observationally equivalent to the standard, space-inefficient semantics.

*This is an extended version of Greenberg [2015], with a great deal of material that does not appear in the conference paper: an exploration of the design space with two other space-efficient calculi and complete proofs.*

## 1 Introduction

Types are an extremely successful form of lightweight specification: programmers can state their intent—e.g., `plus` is a function that takes two numbers and returns another number—and then type checkers can ensure that a program conforms to the programmers intent. Types can only go so far though: division is, like addition, a function that takes two numbers and returns another number... so long as the second number isn’t zero. Conventional type systems do a good job of stopping many kinds of errors, but most type systems cannot protect partial operations like division and array indexing. Advanced techniques—singleton and dependent types, for example—can cover many of these cases, allowing programmers to use types like “non-zero number” or “index within bounds” to specify the domains on which partial operations are safe. Such techniques are demanding: they can be difficult to understand, they force certain programming idioms, and they place heavy constraints on the programming language, requiring purity or even strong normalization.

*Contracts* are a popular compromise: programmers write type-like contracts of the form  $\text{Int} \rightarrow \{x:\text{Int} \mid x \neq 0\} \rightarrow \text{Int}$ , where the predicates  $x \neq 0$  are written in code. These type-like specifications can then be checked at runtime Findler and Felleisen [2002]. Models of contract calculi have taken two forms: latent and manifest Greenberg et al. [2012]. We take the manifest approach here, which means checking contracts with *casts*, written  $\langle T_1 \Rightarrow T_2 \rangle^l e$ . Checking a *predicate contract* (also called a *refinement type*, though that term is overloaded) like  $\{x:\text{Int} \mid x \neq 0\}$  on a number  $n$  involves running the predicate  $x \neq 0$  with  $n$  for  $x$ . Casts from one predicate contract to another,  $\langle \{x:B \mid e_1\} \Rightarrow \{x:B \mid e_2\} \rangle^l$ , take a constant  $k$  and check to see that  $e_2[k/x] \rightarrow^* \text{true}$ . It’s hard to know what to do with function casts at runtime: in  $\langle T_{11} \rightarrow T_{12} \Rightarrow T_{21} \rightarrow T_{22} \rangle^l e$ , we know that  $e$  is a  $T_{11} \rightarrow T_{12}$ , but what does that tell us about treating  $e$  as a  $T_{21} \rightarrow T_{22}$ ? Findler and Felleisen’s insight is that we must defer checking, waiting until the cast value  $e$  gets an argument Findler and Felleisen [2002]. These deferred checks are recorded on the value by means of a *function proxy*, i.e.,  $\langle T_{11} \rightarrow T_{12} \Rightarrow T_{21} \rightarrow T_{22} \rangle^l e$  is a value when  $e$  is a value; applying a function proxy unwraps it contravariantly. We check the domain contract  $T_1$  on  $e$ , run the original function  $f$  on the result, and then check that result against the codomain contract  $T_2$ :

$$\begin{aligned} & \langle \langle T_{11} \rightarrow T_{12} \Rightarrow T_{21} \rightarrow T_{22} \rangle^l e_1 \rangle e_2 \rightarrow \\ & \langle T_{12} \Rightarrow T_{22} \rangle^l (e_1 (\langle T_{21} \Rightarrow T_{11} \rangle^l e_2)) \end{aligned}$$

Findler and Felleisen neatly designed a system for contract checking in a higher-order world, but there is a problem: contract checking is space inefficient Herman et al. [2007].

Contract checking’s space inefficiency can be summed up as follows: **function proxies break tail calls**. Calls to an unproxied function from a tail position can be optimized to not allocate stack frames. Proxied functions, however, will unwrap to have codomain contracts—breaking tail calls. We discuss other sources of space inefficiency below, but breaking tail calls is the most severe. Consider factorial written in accumulator passing style. The developer may believe that the following can be compiled to use tail calls:

$$\begin{aligned} \text{fact} &: \{x:\text{Int} \mid x \geq 0\} \rightarrow \{x:\text{Int} \mid x \geq 0\} \rightarrow \{x:\text{Int} \mid x \geq 0\} \\ &= \lambda x:\{x:\text{Int} \mid \text{true}\}. \lambda y:\{y:\text{Int} \mid \text{true}\}. \\ &\quad \text{if } x = 0 \text{ then } y \text{ else fact } (x - 1) (x * y) \end{aligned}$$

A cast insertion algorithm Swamy et al. [2009] might produce the following non-tail recursive function:

$$\begin{aligned} \text{fact} &= \\ &\langle \{x:\text{Int} \mid \text{true}\} \rightarrow \{y:\text{Int} \mid \text{true}\} \rightarrow \{z:\text{Int} \mid \text{true}\} \Rightarrow \\ &\quad \{x:\text{Int} \mid x \geq 0\} \rightarrow \{y:\text{Int} \mid y \geq 0\} \rightarrow \{z:\text{Int} \mid x \geq 0\} \rangle^{l_{\text{fact}}} \\ &\lambda x:\{x:\text{Int} \mid \text{true}\}. \lambda y:\{y:\text{Int} \mid \text{true}\}. \\ &\quad \text{if } x = 0 \text{ then } y \text{ else} \\ &\quad (\langle \{x:\text{Int} \mid x \geq 0\} \Rightarrow \{x:\text{Int} \mid \text{true}\} \rangle^{l_{\text{fact}}} (\text{fact } \dots)) \end{aligned}$$

Tail-call optimization is essential for usable functional languages—we believe that space inefficiency has been one of two significant obstacles for pervasive use of higher-order contract checking. (The other is state, which we do not treat here.)

In this work, we show how to achieve semantics-preserving space efficiency for non-dependent contract checking. Our approach is inspired by work on *gradual typing* Siek and Taha [2006], a form of (manifest) contracts designed to mediate dynamic and simple typing—that is, gradual typing (a) allows the dynamic type, and (b) restricts the predicates in contracts to checks on type tags. Herman et al. [2007] developed the first space-efficient gradually typed system, using Henglein’s coercions Henglein [1994]; Siek and Wadler [2010] devised a related system supporting blame. The essence of the solution is to allow casts to merge: given two adjacent casts  $\langle T_2 \Rightarrow T_3 \rangle^{l_2} (\langle T_1 \Rightarrow T_2 \rangle^{l_1} e)$ , we must somehow combine them into a single cast. Siek and Wadler annotate their casts with an intermediate type representing the greatest lower bound of the types encountered. Such a trick doesn’t work in our more general setting: simple types plus dynamic form a straightforward lattice using type precision as the ordering, but it’s less clear what to do when we have arbitrary predicate contracts.

We offer three *modes* of space-efficiency; all of the modes are defined in a single calculus which we call  $\lambda_H$ . Each mode enjoys varying levels of soundness with respect to the standard, space-inefficient semantics of classic  $\lambda_H$ . We sketch here the mode-indexed rules for combining annotations on casts—the key rules for space efficiency.

The *forgetful* mode uses empty annotations,  $\bullet$ ; we combine two casts by dropping intermediate types:

$$\langle T_2 \xRightarrow{\bullet} T_3 \rangle^{l_2} (\langle T_1 \xRightarrow{\bullet} T_2 \rangle^{l_1} e) \rightarrow_F \langle T_1 \xRightarrow{\bullet} T_3 \rangle^{l_2} e$$

Surprisingly, this evaluation rule is type safe and somewhat sound with respect to the classic mode, as discovered by Greenberg [2013]: if classic  $\lambda_H$  produces a value, so does forgetful  $\lambda_H$ .

The *heedful* mode uses sets of types  $S_i$  as its annotations, making sure to save the intermediate type:

$$\langle T_2 \xRightarrow{S_2} T_3 \rangle^{l_2} (\langle T_1 \xRightarrow{S_1} T_2 \rangle^{l_1} e) \rightarrow_H \langle T_1 \xRightarrow{S_1 \cup S_2 \cup \{T_2\}} T_3 \rangle^{l_2} e$$

In Siek and Wadler’s terms, we use the powerset lattice for annotations, while they use pointed types. Heedful and classic  $\lambda_H$  are almost identical, except sometimes they blame different labels.

Finally, the *eidetic* mode annotates casts with *refinement lists* and *function coercions*—a new form of coercion inspired by Greenberg [2013]. The coercions keep track of checking so well that the type indices and blame labels on casts are unnecessary:

$$\langle T_2 \xRightarrow{c_2} T_3 \rangle^{\bullet} (\langle T_1 \xRightarrow{c_1} T_2 \rangle^{\bullet} e) \rightarrow_E \langle T_1 \xRightarrow{c_1 \triangleright c_2} T_3 \rangle^{\bullet} e$$

These coercions form a skew lattice: refinement lists have ordering constraints that break commutativity. Eidetic  $\lambda_H$  is space efficient and observationally equivalent to the classic mode.

Since eidetic and classic  $\lambda_H$  behave the same, why bother with forgetful and heedful? First and foremost, the calculi offer insights into the semantics of contracts: the soundness of forgetful  $\lambda_H$  depends on a certain philosophy of contracts; heedful  $\lambda_H$  relates to threesomes without blame Siek and Wadler [2010]. Second, we offer them as alternative points in the design space. Finally and perhaps cynically, they are strawmen—warm up exercises for eidetic  $\lambda_H$ .

We claim two contributions:

1. Eidetic  $\lambda_H$  is the first manifest contract calculus that is both *sound* and *space efficient* with respect to the classic semantics—a result contrary to Greenberg [2013], who conjectured that such a result is impossible. We believe that space efficiency is a critical step towards the implementation of practical languages with manifest contracts.
2. A framework for defining space efficiency in manifest contract systems, with an exploration of the design space. We identify common structures and methods in the operational semantics as well as in the proofs of type soundness, soundness with regard to the classic framework, and space bounds.

We do not prove a blame theorem Wadler and Findler [2009], since we lack the clear separation of dynamic and static typing found in gradual typing. We conjecture that such a theorem could be proved for classic and eidetic  $\lambda_H$ —but perhaps not for forgetful and heedful  $\lambda_H$ , which skip checks and change blame labels. Our model has two limits worth mentioning: we do not handle dependency, a common and powerful feature in manifest systems; and, our bounds for space efficiency are *galactic*—they establish that contracts consume constant space, but do nothing to reduce that constant Lipton [2010]. Our contribution is showing that sound space efficiency is *possible* where it was believed to be impossible Greenberg [2013]; we leave evidence that it is *practicable* for future work.

Our proofs are available in the extended version Greenberg [2014], Appendices A–C.

Readers who are very familiar with this topic can read Figures 1, 2, and 4 and then skip directly to Section 3.5. Readers who understand the space inefficiency of contracts but aren’t particularly familiar with manifest contracts can skip Section 2 and proceed to Section 3.

## 2 Function proxies

Space inefficient contract checking breaks tail recursion—a showstopping problem for realistic implementations of pervasive contract use. PLT Racket’s contract system PLT [b], the most widely used higher-order contract system, takes a “macro” approach to contracts: contracts typically appear only on module interfaces, and aren’t checked within a module. Their approach comes partly out of a philosophy of breaking invariants inside modules but not out of them, but also partly out of a need to retain tail recursion within modules. Space inefficiency has shaped the way their contract system has developed. They do not use our “micro” approach, wherein annotations and casts permeate the code.

Tail recursion aside, there is another important source of space inefficiency: the unbounded number of function proxies. Hierarchies of libraries are a typical example: consider a list library and a set library built using increasingly sorted lists. We might have:

$$\begin{aligned} \text{null} &: \alpha \text{ List} \rightarrow \{x:\text{Bool} \mid \text{true}\} &= \dots \\ \text{head} &: \{x:\alpha \text{ List} \mid \text{not}(\text{null } x)\} \rightarrow \alpha &= \dots \\ \text{empty} &: \alpha \text{ Set} \rightarrow \{x:\text{Bool} \mid \text{true}\} &= \text{null} \\ \text{min} &: \{x:\alpha \text{ Set} \mid \text{not}(\text{empty } x)\} \rightarrow \alpha &= \text{head} \end{aligned}$$

Our code reuse comes with a price: even though the precondition on `min` is effectively the same as that on `head`, we must have two function proxies, and the non-emptiness of the list representing the set is checked twice: first by checking `empty`, and again by checking `null` (which is the same function). Blame systems like those in PLT Racket encourage modules to redeclare contracts to avoid being blamed—which can result in redundant checking.

Or consider a library of drawing primitives based around painters, functions of type  $\text{Canvas} \rightarrow \text{Canvas}$ . An underlying graphics library offers basic functions for manipulating canvases and functions over canvases, e.g., `primFlipH` is a painter transformer—of type  $(\text{Canvas} \rightarrow \text{Canvas}) \rightarrow (\text{Canvas} \rightarrow \text{Canvas})$ —that flips the generated images horizontally. A wrapper library may add derived functions while re-exporting the underlying functions with refinement types specifying a square canvas dimensions, where  $\text{SquareCanvas} = \{x:\text{Canvas} \mid \text{width}(x) = \text{height}(x)\}$ :

$$\begin{aligned} \text{flipH } p &= \langle \text{Canvas} \rightarrow \text{Canvas} \Rightarrow \\ &\quad \text{SquareCanvas} \rightarrow \text{SquareCanvas} \rangle^l \\ &\quad (\text{primFlipH} \\ &\quad (\langle \text{SquareCanvas} \rightarrow \text{SquareCanvas} \Rightarrow \\ &\quad \quad \text{Canvas} \rightarrow \text{Canvas} \rangle^l p)) \end{aligned}$$

The wrapper library only accepts painters with appropriately refined types, but must strip away these refinements before calling the underlying implementation—which demands  $\text{Canvas} \rightarrow \text{Canvas}$  painters. The wrapper library then

has to cast these modified functions *back* to the refined types. Calling `flipH (flipH p)` yields:

$$\begin{aligned} & \langle \text{Canvas} \rightarrow \text{Canvas} \Rightarrow \text{SquareCanvas} \rightarrow \text{SquareCanvas} \rangle^l \\ & (\text{primFlipH} \\ & \quad (\langle \text{SquareCanvas} \rightarrow \text{SquareCanvas} \Rightarrow \text{Canvas} \rightarrow \text{Canvas} \rangle^l \\ & \quad \quad (\langle \text{Canvas} \rightarrow \text{Canvas} \Rightarrow \text{SquareCanvas} \rightarrow \text{SquareCanvas} \rangle^l \\ & \quad \quad \quad (\text{primFlipH} \\ & \quad \quad \quad \quad (\langle \text{SquareCanvas} \rightarrow \text{SquareCanvas} \Rightarrow \\ & \quad \quad \quad \quad \quad \text{Canvas} \rightarrow \text{Canvas} \rangle^l p)))))) \end{aligned}$$

That is, we first cast  $p$  to a plain painter and return a new painter  $p'$ . We then cast  $p'$  into and then immediately out of the refined type, before continuing on to flip  $p'$ . All the while, we are accumulating many function proxies beyond the wrapping that the underlying implementation of `primFlipH` is doing. A space-efficient scheme for manifest contracts bounds the number of function proxies that can accumulate. Redundant wrapping can become quite extreme, especially for continuation-passing programs. Function proxies are the essential problem: nothing bounds their accumulation. Unfolding unboundedly many function proxies creates stacks of unboundedly many checks—which breaks tail calls. If we can have a constant number of function proxies that produce stacks of checks of a constant size, then we can have tail call optimization.

We can also adapted Herman et al.’s mutually recursive even and odd functions. We might write them with type ascriptions as follows, in what appears to be a tail recursive program consuming constant space:

$$\begin{aligned} \text{odd} &= \lambda x:\{x:\text{Int} \mid \text{true}\}. \text{ if } (x = 0) \\ &\quad \text{then false} : \{b:\text{Bool} \mid b \vee (x \bmod 2 = 0)\} \\ &\quad \text{else even } (x - 1) \\ \text{even} &: \{x:\text{Int} \mid \text{true}\} \rightarrow \{b:\text{Bool} \mid b \vee (x - 1 \bmod 2 = 0)\} = \\ &\quad \lambda x:\{x:\text{Int} \mid \text{true}\}. \text{ if } (x = 0) \text{ then true else odd } (x - 1) \end{aligned}$$

A cast insertion algorithm Swamy et al. [2009] might produce:

$$\begin{aligned} \text{odd} &= \lambda x:\{x:\text{Int} \mid \text{true}\}. \text{ if } (x = 0) \\ &\quad \text{then } \langle \{b:\text{Bool} \mid \text{true}\} \Rightarrow \\ &\quad \quad \{b:\text{Bool} \mid b \vee (x \bmod 2 = 0)\} \rangle^{l_1} \text{ false} \\ &\quad \text{else even } (x - 1) \\ \text{even} &= \langle \{x:\text{Int} \mid \text{true}\} \rightarrow \{b:\text{Bool} \mid \text{true}\} \Rightarrow \\ &\quad \{x:\text{Int} \mid \text{true}\} \rightarrow \\ &\quad \quad \{b:\text{Bool} \mid b \vee ((x + 1) \bmod 2 = 0)\} \rangle^{l_2} \\ &\quad (\lambda x:\{x:\text{Int} \mid \text{true}\}. \text{ if } (x = 0) \\ &\quad \quad \text{then true} \\ &\quad \quad \text{else } \langle \{b:\text{Bool} \mid b \vee (x - 1 \bmod 2 = 0)\} \Rightarrow \\ &\quad \quad \quad \{b:\text{Bool} \mid \text{true}\} \rangle^{l_3} (\text{odd } (x - 1))) \end{aligned}$$

Note that the casts labeled  $l_2$  and  $l_3$  both break tail recursion: the former leads to unwrapping whenever `even` is called, while the latter moves the call to `odd` from tail position. Breaking tail-call optimization is very bad: we went from constant space to *linear* space. That is, inserting contract checks can change a program’s asymptotic space efficiency!

### 3 Classic manifest contracts

The standard manifest contract calculus,  $\lambda_H$ , is originally due to Flanagan Flanagan [2006]. We give the syntax for the non-dependent fragment in Figure 1. We have highlighted in **yellow** the four syntactic forms relevant to contract checking. This paper paper discusses four modes of  $\lambda_H$ : classic  $\lambda_H$ , mode C; forgetful  $\lambda_H$ , mode F; heedful  $\lambda_H$ , mode H; and eidetic  $\lambda_H$ , mode E. Each of these languages uses the syntax of Figure 1, while the typing rules and operational semantics are indexed by the mode  $m$ . The proofs and metatheory are also mode-indexed. In an extended version of this work, we develop two additional modes with slightly different properties from eidetic  $\lambda_H$ , filling out a “framework” for space-efficient manifest contracts Greenberg [2014]. We omit the other two modes here to save space for eidetic  $\lambda_H$ , which is the only mode that is sound with respect to classic  $\lambda_H$ . We summarize how each of these modes differ in Section 3.2—but first we (laconically) explain the syntax and the interesting bits of the operational semantics.

<b>Modes</b>	
$m$	$::=$ C    classic $\lambda_H$ ; Section 3
	F    forgetful $\lambda_H$ ; Section 4
	H    heedful $\lambda_H$ ; Section 5
	E    eidetic $\lambda_H$ ; Section 6
<b>Types</b>	
$B$	$::=$ Bool   ...
$T$	$::=$ $\{x:B \mid e\}$   $T_1 \rightarrow T_2$
<b>Terms</b>	
$e$	$::=$ $x$   $k$   $\lambda x:T. e$   $e_1 e_2$   $op(e_1, \dots, e_n)$
	$\langle T_1 \xrightarrow{a} T_2 \rangle^l e$   $\langle \{x:B \mid e_1\}, e_2, k \rangle^l$   $\uparrow l$
	$\langle \{x:B \mid e_1\}, s, r, k, e \rangle^\bullet$
<b>Annotations: type set, coercions, and refinement lists</b>	
$a$	$::=$ $\bullet$   $S$   $c$
$S$	$::=$ $\emptyset$   $\{T_1, \dots, T_n\}$
$c$	$::=$ $r$   $c_1 \mapsto c_2$
$r$	$::=$ nil   $\{x:B \mid e\}^l, r$
<b>Statuses</b>	
$s$	$::=$ $\checkmark$   $?$
<b>Locations</b>	
$l$	$::=$ $\bullet$   $l_1$   ...

Figure 1: Syntax of  $\lambda_H$

The metavariable  $B$  is used for base types, of which at least **Bool** must be present. There are two kinds of types. First, *predicate contracts*  $\{x:B \mid e\}$ , also called *refinements of base types* or just *refinement types*, denotes constants  $k$  of base type  $B$  such that  $e[k/x]$  holds—that is, such that  $e[k/x] \rightarrow_m^* \text{true}$  for any mode  $m$ . Function types  $T_1 \rightarrow T_2$  are standard.

The terms of  $\lambda_H$  are largely those of the simply-typed lambda calculus: variables, constants  $k$ , abstractions, applications, and operations should all be familiar. The first distinguishing feature of  $\lambda_H$ ’s terms is the *cast*, written  $\langle T_1 \xrightarrow{a} T_2 \rangle^l e$ . Here  $e$  is term of type  $T_1$ ; the cast checks whether  $e$  can be treated as a  $T_2$ —if  $e$  doesn’t cut it, the cast will use its label  $l$  to raise the uncatchable exception  $\uparrow l$ , read “blame  $l$ ”. Our casts also have annotations  $a$ . Classic and forgetful  $\lambda_H$  don’t need annotations—we write  $\bullet$ . Heedful  $\lambda_H$  uses type sets  $S$  to track space-efficiently pending checks. Eidetic  $\lambda_H$  uses coercions  $c$ , based on coercions in Henglein [1994]. We explain coercions in greater detail in Section 6, but they amount to lists of blame-annotated refinement types  $r$  and function coercions.

The three remaining forms—active checks, blame, and coercion stacks—only occur as the program evaluates. Casts between refinement types are checked by *active checks*  $\langle \{x:B \mid e_1\}, e_2, k \rangle^l$ . The first term is the type being checked—necessary for the typing rule. The second term is the current status of the check; it is an invariant that  $e_1[k/x] \rightarrow_m^* e_2$ . The final term is the constant being checked, which is returned wholesale if the check succeeds. When checks fail, the program raises *blame*, an uncatchable exception written  $\uparrow l$ . A *coercion stack*  $\langle \{x:B \mid e_1\}, s, r, k, e \rangle^\bullet$  represents the state of checking a coercion; we only use it in eidetic  $\lambda_H$ , so we postpone discussing it until Section 6.

### 3.1 Core operational semantics

Our mode-indexed operational semantics for our manifest calculi comprise three relations:  $\text{val}_m e$  identifies terms that are values in mode  $m$  (or  $m$ -values),  $\text{result}_m e$  identifies  $m$ -results, and  $e_1 \rightarrow_m e_2$  is the small-step reduction relation for mode  $m$ . It is more conventional to fix values as a syntactic subset, but that approach would be confusing here: we would need three different metavariables to unambiguously refer to values from each language. The mode-indexed value and result relations neatly avoid any potential confusion between metavariables. Each mode defines its own value rule for function proxies. Figure 2 defines the core rules. The rules for classic  $\lambda_H$  ( $m = C$ ) are in **salmon**; the shared space-efficient rules are in **periwinkle**. To save space, we pass over standard rules.

The mode-agnostic value rules are straightforward: constants are always values (**V\_CONST**), as are lambdas (**V\_ABS**). Each mode defines its own value rule for function proxies, **V\_PROXY** $m$ . The classic rule, **V\_PROXYC**, says that a function proxy

$$\langle T_{11} \rightarrow T_{12} \xrightarrow{\bullet} T_{21} \rightarrow T_{22} \rangle^l e$$

is a C-value when  $e$  is a C-value. That is, function proxies can wrap lambda abstractions and other function proxies alike. Other modes only allow lambda abstractions to be proxied while requiring that the annotations are appropriate.

Values and results

$\boxed{\text{val}_m e}$

$\boxed{\text{result}_m e}$

$$\frac{}{\text{val}_m k} \text{ V\_CONST}$$

$$\frac{}{\text{val}_m \lambda x:T. e} \text{ V\_ABS}$$

$$\frac{\text{val}_C e}{\text{val}_C \langle T_{11} \rightarrow T_{12} \xRightarrow{\bullet} T_{21} \rightarrow T_{22} \rangle^l e} \text{ V\_PROXYC}$$

$$\frac{\text{val}_m e}{\text{result}_m e} \text{ R\_VAL}$$

$$\frac{}{\text{result}_m \uparrow l} \text{ R\_BLAME}$$

Shared operational semantics

$\boxed{e_1 \rightarrow_m e_2}$

$$\frac{\text{val}_m e_2}{(\lambda x:T. e_{12}) e_2 \rightarrow_m e_{12}[e_2/x]} \text{ E\_BETA}$$

$$\frac{\text{val}_m e_1 \dots \text{val}_m e_n}{op(e_1, \dots, e_n) \rightarrow_m \llbracket op \rrbracket(e_1, \dots, e_n)} \text{ E\_OP}$$

$$\frac{\text{val}_m \langle T_{11} \rightarrow T_{12} \xRightarrow{a} T_{21} \rightarrow T_{22} \rangle^l e_1 \quad \text{val}_m e_2}{(\langle T_{11} \rightarrow T_{12} \xRightarrow{a} T_{21} \rightarrow T_{22} \rangle^l e_1) e_2 \rightarrow_m \langle T_{12} \xRightarrow{\text{cod}(a)} T_{22} \rangle^l (e_1 (\langle T_{21} \xRightarrow{\text{dom}(a)} T_{11} \rangle^l e_2))} \text{ E\_UNWRAP}$$

$$\begin{aligned} \text{dom}(\bullet) &= \bullet \\ \text{dom}(S) &= \bigcup_{T \in \mathcal{S}} \text{dom}(T) \\ \text{dom}(c_1 \mapsto c_2) &= c_1 \end{aligned}$$

$$\begin{aligned} \text{cod}(\bullet) &= \bullet \\ \text{cod}(S) &= \bigcup_{T \in \mathcal{S}} \text{cod}(T) \\ \text{cod}(c_1 \mapsto c_2) &= c_2 \end{aligned}$$

$$\frac{}{\langle \{x:B \mid e_1\} \xRightarrow{\bullet} \{x:B \mid e_2\} \rangle^l k \rightarrow_C \langle \{x:B \mid e_2\}, e_2[k/x], k \rangle^l} \text{ E\_CHECKNONEC}$$

$$\frac{}{\langle \{x:B \mid e\}, \text{true}, k \rangle^l \rightarrow_m k} \text{ E\_CHECKOK}$$

$$\frac{}{\langle \{x:B \mid e\}, \text{false}, k \rangle^l \rightarrow_m \uparrow l} \text{ E\_CHECKFAIL}$$

$$\frac{e_1 \rightarrow_m e'_1}{e_1 e_2 \rightarrow_m e'_1 e_2} \text{ E\_APPL}$$

$$\frac{\text{val}_m e_1 \quad e_2 \rightarrow_m e'_2}{e_1 e_2 \rightarrow_m e_1 e'_2} \text{ E\_APPR}$$

$$\frac{\text{val}_m e_1 \dots \text{val}_m e_{i-1} \quad e_i \rightarrow_m e'_i}{op(e_1, \dots, e_{i-1}, e_i, \dots, e_n) \rightarrow_m op(e_1, \dots, e_{i-1}, e'_i, \dots, e_n)} \text{ E\_OPINNER}$$

$$\frac{e \rightarrow_C e'}{\langle T_1 \xRightarrow{\bullet} T_2 \rangle^l e \rightarrow_C \langle T_1 \xRightarrow{\bullet} T_2 \rangle^l e'} \text{ E\_CASTINNERC}$$

$$\frac{e_2 \rightarrow_m e'_2}{\langle \{x:B \mid e_1\}, e_2, k \rangle^l \rightarrow_m \langle \{x:B \mid e_1\}, e'_2, k \rangle^l} \text{ E\_CHECKINNER}$$

$$\frac{e_2 \rightarrow_E e'_2 \quad e_2 \neq \langle T_1 \xRightarrow{a'} T_2 \rangle^{l'} e''_2}{\langle T_2 \xRightarrow{a} T_3 \rangle^l e_2 \rightarrow_E \langle T_2 \xRightarrow{a} T_3 \rangle^l e'_2} \text{ E\_CASTINNERE}$$

$$\frac{a_3 = \text{merge}_E(T_1, a_1, T_2, a_2, T_3)}{\langle T_2 \xRightarrow{a_2} T_3 \rangle^l (\langle T_1 \xRightarrow{a_1} T_2 \rangle^{l'} e_2) \rightarrow_E \langle T_1 \xRightarrow{a_3} T_3 \rangle^l e_2} \text{ E\_CASTMERGEE}$$

$$\frac{}{\uparrow l e_2 \rightarrow_m \uparrow l} \text{ E\_APPRAISEL}$$

$$\frac{\text{val}_m e_1}{e_1 \uparrow l \rightarrow_m \uparrow l} \text{ E\_APPRAISER}$$

$$\frac{}{\langle T_1 \xRightarrow{S} T_2 \rangle^l \uparrow l' \rightarrow_m \uparrow l'} \text{ E\_CASTRAISE}$$

$$\frac{\text{val}_m e_1 \dots \text{val}_m e_{i-1}}{op(e_1, \dots, e_{i-1}, \uparrow l, \dots, e_n) \rightarrow_m \uparrow l} \text{ E\_OPRAISE}$$

$$\frac{}{\langle \{x:B \mid e\}, \uparrow l, k \rangle^{l'} \rightarrow_m \uparrow l} \text{ E\_CHECKRAISE}$$

Figure 2: Core operational semantics of  $\lambda_H$ ; classic  $\lambda_H$  rules are salmon; space-efficient rules are periwinkle

All of the space-efficient calculi in the literature take our approach, where a function cast applied to a value *is* a value; some space inefficient ones do, too Findler and Felleisen [2002], Gronski and Flanagan [2007], Greenberg et al. [2012]. In other formulations of  $\lambda_H$  in the literature, function proxies are implemented by introducing a new lambda as a wrapper à la Findler and Felleisen’s  $\overline{wrap}$  operator Findler and Felleisen [2002], Flanagan [2006], Siek and Taha [2006], Belo et al. [2011]. Such an  $\eta$ -expansion semantics is convenient, since then applications only ever reduce by  $\beta$ -reduction. But it wouldn’t suit our purposes at all: space efficiency demands that we combine function proxies. We can imagine a third, ungainly semantics that looks into closures rather than having explicit function proxies. Results don’t depend on the mode:  $m$ -values are always  $m$ -results (R\_VAL); blame is always a result, too (R\_BLAKE).

E\_BETA applies lambda abstractions via substitution, using a call-by-value rule. Note that  $\beta$  reduction in mode  $m$  requires that the argument is an  $m$ -value. The reduction rule for operations (E\_OP) defers to operations’ denotations,  $\llbracket op \rrbracket$ ; since these may be partial (e.g., division), we assign types to operations that guarantee totality (see Section 3.3). That is, partial operations are a potential source of stuckness, and the types assigned to operations must guarantee the absence of stuckness. Robin Milner famously stated that “well typed expressions don’t go wrong” Milner [1978]; his programs could go wrong by (a) applying a boolean like a function or (b) conditioning on a function like a boolean. Systems with more base types can go wrong in more ways, some of which are hard to capture in standard type systems. Contracts allow us to bridge that gap. Letting operations get stuck is a philosophical stance—contracts expand the notion of “wrong”—that supports our forgetful semantics (Section 4).

E\_UNWRAP applies function proxies to values, contravariantly in the domain and covariantly in the codomain. We also split up each cast’s annotation, using  $\text{dom}(a)$  and  $\text{cod}(a)$ . E\_CHECKNONE returns a cast between refinement types into an active check with the same blame label. We discard the source type—we already know that  $k$  is a  $\{x:B \mid e_1\}$ —and substitute the scrutinee into the target type,  $e_2[k/x]$ , as the current state of checking. We must also hold onto the scrutinee, in case the check succeeds. We are careful to not apply this rule in heedful and eidetic modes, which must generate annotations before running checks; we discuss these more in those modes’ sections. Active checks evaluate by the congruence rule E\_CHECKINNER until one of three results adheres: the predicate returns **true**, so the whole active check returns the scrutinee (E\_CHECKOK); the predicate returns **false**, so the whole active check raises blame using the label on the check (E\_CHECKFAIL); or blame was raised during checking, and we propagate it via E\_CHECKRAISE. heedful and eidetic use slightly different forms, described in their respective sections.

The core semantics includes several other congruence rules: E\_APPL, E\_APPR, and E\_OPINNER. Since space bounds rely not only on limiting the number of function proxies but also on accumulation of casts on the stack, the core semantics doesn’t include a cast congruence rule. The congruence rule for casts in classic  $\lambda_H$ , E\_CASTINNERC, allows for free use of congruence. In the space-efficient calculi, the use of congruence is instead limited by the rules E\_CASTINNERE and E\_CASTMERGEE. Cast arguments only take congruent steps when they aren’t casts themselves. A cast applied to another cast *merges*, using the **merge** function. Each space-efficient calculus uses a different annotation scheme, so each one has a different merge function. We deliberately leave **merge** undefined sometimes—heedful and eidetic  $\lambda_H$  must control when E\_CASTMERGEE can apply. Note that we don’t need to specify  $m \neq C$  in E\_CASTMERGEE—we just don’t define a merge operator for classic  $\lambda_H$ . We have E\_CASTMERGEE arbitrarily retain the label of the outer cast. No choice is “right” here—we discuss this issue further in Section 5. In addition to congruence rules, there are blame propagation rules, which are universal: E\_APPRAISEL, E\_APPRAISER, E\_CASTRAISE, E\_OPRAISE. These rules propagate the uncatchable exception  $\uparrow l$  while obeying call-by-value rules.

### 3.2 Cast merges by example

Each mode’s section explains its semantics in detail, but we can summarize the cast merging rules here by example. Consider the following term:

$$e = \langle \{x:\text{Int} \mid x \bmod 2 = 0\} \xrightarrow{\bullet} \{x:\text{Int} \mid x \neq 0\} \rangle^{l_3} \\ ((\{x:\text{Int} \mid x \geq 0\} \xrightarrow{\bullet} \{x:\text{Int} \mid x \bmod 2 = 0\})^{l_2} \\ ((\{x:\text{Int} \mid \text{true}\} \xrightarrow{\bullet} \{x:\text{Int} \mid x \geq 0\})^{l_1} - 1))$$

Here  $e$  runs three checks on integer  $-1$ : first for non-negativity (blaming  $l_1$  on failure), then for evenness (blaming  $l_2$  on failure), and then for non-zerosness (blaming  $l_3$  on failure). Classic and eidetic  $\lambda_H$  both blame  $l_1$ ; heedful  $\lambda_H$  also raises blame, though it blames a different label,  $l_3$ ; forgetful  $\lambda_H$  actually *accepts* the value, returning  $-1$ . We discuss the operational rules for modes other than C in detail in each mode’s section; for now, we repeat the derived rules for merging casts from Section 1.

Classic  $\lambda_H$  evaluates the casts step-by-step: first it checks whether  $-1$  is positive, which fails, so  $e \rightarrow_C^* \uparrow l_1$  (see Figure 3). We first step by E\_CHECKNONE, starting checking at the innermost cast. Using congruence rules, we run E\_OP to reduce the contract’s predicate, finding **false**. E\_CHECKFAIL then raises blame and E\_CASTRAISE propagates it. Forgetful  $\lambda_H$  doesn’t use annotations at all—it just forgets the intermediate casts, effectively using the following

$$\begin{aligned}
e &= \langle \{x:\text{Int} \mid x \bmod 2 = 0\} \dot{\Rightarrow} \{x:\text{Int} \mid x \neq 0\} \rangle^{l_3} \\
&\quad ((\langle \{x:\text{Int} \mid x \geq 0\} \dot{\Rightarrow} \{x:\text{Int} \mid x \bmod 2 = 0\} \rangle^{l_2} \\
&\quad (\langle \{x:\text{Int} \mid \text{true}\} \dot{\Rightarrow} \{x:\text{Int} \mid x \geq 0\} \rangle^{l_1} -1)) \\
&\quad (\text{E\_CASTINNERC/E\_CHECKNONE})) \\
&\rightarrow_C \langle \{x:\text{Int} \mid x \geq 0\} \dot{\Rightarrow} \{x:\text{Int} \mid x \neq 0\} \rangle^{l_3} \\
&\quad ((\langle \{x:\text{Int} \mid x \geq 0\} \dot{\Rightarrow} \{x:\text{Int} \mid x \bmod 2 = 0\} \rangle^{l_2} \\
&\quad \langle \{x:\text{Int} \mid x \geq 0\}, -1 \geq 0, -1 \rangle^{l_1}) \\
&\quad (\text{E\_CASTINNERC/E\_CHECKINNER/E\_OP})) \\
&\rightarrow_C \dots \langle \{x:\text{Int} \mid x \geq 0\}, \text{false}, -1 \rangle^{l_1} \\
&\quad (\text{E\_CASTINNERC/E\_CHECKFAIL}) \\
&\rightarrow_C \dots \uparrow l_1 \\
&\rightarrow_C^* \uparrow l_1 \quad (\text{E\_CASTRAISE})
\end{aligned}$$

Figure 3: An example of classic  $\lambda_H$

rule:

$$\langle T_2 \dot{\Rightarrow} T_3 \rangle^{l_2} (\langle T_1 \dot{\Rightarrow} T_2 \rangle^{l_1} e) \rightarrow_F \langle T_1 \dot{\Rightarrow} T_3 \rangle^{l_2} e$$

It never checks for non-negativity or evenness, skipping straight to the check that  $-1$  is non-zero.

$$\begin{aligned}
e &\rightarrow_F \langle \{x:\text{Int} \mid x \geq 0\} \dot{\Rightarrow} \{x:\text{Int} \mid x \neq 0\} \rangle^{l_3} \\
&\quad ((\langle \{x:\text{Int} \mid \text{true}\} \dot{\Rightarrow} \{x:\text{Int} \mid x \geq 0\} \rangle^{l_1} -1)) \\
&\rightarrow_F \langle \{x:\text{Int} \mid \text{true}\} \dot{\Rightarrow} \{x:\text{Int} \mid x \neq 0\} \rangle^{l_3} -1 \\
&\rightarrow_F^* -1
\end{aligned}$$

Heedful  $\lambda_H$  works by annotating casts with a set of intermediate types, effectively using the rule:

$$\langle T_2 \xrightarrow{S_2} T_3 \rangle^{l_2} (\langle T_1 \xrightarrow{S_1} T_2 \rangle^{l_1} e) \rightarrow_H \langle T_1 \xrightarrow{S_1 \cup S_2 \cup \{T_2\}} T_3 \rangle^{l_2} e$$

Every type in a type set needs to be checked, but the order is essentially nondeterministic: heedful  $\lambda_H$  checks that  $-1$  is non-negative and even in *some* order. Whichever one is checked first fails; both cases raise  $\uparrow l_3$ .

$$\begin{aligned}
e &\rightarrow_H^* \langle \{x:\text{Int} \mid x \geq 0\} \xrightarrow{\{\{x:\text{Int} \mid x \bmod 2 = 0\}\}} \{x:\text{Int} \mid x \neq 0\} \rangle^{l_3} \\
&\quad ((\langle \{x:\text{Int} \mid \text{true}\} \xrightarrow{\bullet} \{x:\text{Int} \mid x \geq 0\} \rangle^{l_1} -1)) \\
&\rightarrow_H^* \langle \{x:\text{Int} \mid \text{true}\} \xrightarrow{S} \{x:\text{Int} \mid x \neq 0\} \rangle^{l_3} -1 \\
&\text{where } S = \{\{x:\text{Int} \mid x \bmod 2 = 0\}, \{x:\text{Int} \mid x \geq 0\}\}
\end{aligned}$$

Finally, eidetic  $\lambda_H$  uses coercions as its annotations; coercions  $c$  are detailed checking plans for running checks in the same order as classic  $\lambda_H$  while skipping redundant checks. As we will see in Section 6, eidetic  $\lambda_H$  generates coercions and then drops blame labels, giving us the rule:

$$\langle T_2 \xrightarrow{c_2} T_3 \rangle^\bullet (\langle T_1 \xrightarrow{c_1} T_2 \rangle^\bullet e) \rightarrow_E \langle T_1 \xrightarrow{c_1 \triangleright c_2} T_3 \rangle^\bullet e$$

There are no redundant checks in the example term  $e$ , so eidetic  $\lambda_H$  does *exactly* the same checking as classic, finding  $e \rightarrow_E^* \uparrow l_1$ .

### 3.3 Type system

All modes share a type system, given in Figure 4. All judgments are universal and simply thread the mode through—except for annotation well formedness  $\vdash_m a \parallel T_1 \Rightarrow T_2$ , which is mode specific, and a single eidetic-specific rule given in Figure 9 in Section 6. The type system comprises several relations: context well formedness  $\vdash_m \Gamma$  and type well formedness  $\vdash_m T$ ; type compatibility  $\vdash T_1 \parallel T_2$ , a mode-less comparison of the *skeleton* of two types; annotation well formedness  $\vdash_m a \parallel T_1 \Rightarrow T_2$ ; and term typing  $\Gamma \vdash_m e : T$ .

Context well formedness is entirely straightforward; type well formedness requires some care to get base types off the ground. We establish as an axiom that the *raw* type  $\{x:B \mid \text{true}\}$  is well formed for every base type  $B$  (WF\_BASE); we then use raw types to check that refinements are well formed:  $\{x:B \mid e\}$  is well formed in mode  $m$  if  $e$  is well typed as a boolean in mode  $m$  when  $x$  is a value of type  $B$  (WF\_REFINE). Without WF\_BASE, WF\_REFINE wouldn't have a well formed context. Function types are well formed in mode  $m$  when their domains



Context and type well formedness

$$\begin{array}{c}
\boxed{\vdash_m \Gamma} \quad \boxed{\vdash_m T} \\
\\
\frac{}{\vdash_m \emptyset} \text{WF\_EMPTY} \qquad \frac{\vdash_m \Gamma \quad \vdash_m T}{\vdash_m \Gamma, x:T} \text{WF\_EXTEND} \\
\\
\frac{}{\vdash_m \{x:B \mid \text{true}\}} \text{WF\_BASE} \quad \frac{x:\{x:B \mid \text{true}\} \vdash_m e : \{x:\text{Bool} \mid \text{true}\}}{\vdash_m \{x:B \mid e\}} \text{WF\_REFINE} \quad \frac{\vdash_m T_1 \quad \vdash_m T_2}{\vdash_m T_1 \rightarrow T_2} \text{WF\_FUN}
\end{array}$$

Type compatibility and annotation well formedness

$$\begin{array}{c}
\boxed{\vdash T_1 \parallel T_2} \quad \boxed{\vdash_m a \parallel T_1 \Rightarrow T_2} \\
\\
\frac{}{\vdash \{x:B \mid e_1\} \parallel \{x:B \mid e_2\}} \text{S\_REFINE} \quad \frac{\vdash T_{11} \parallel T_{21} \quad \vdash T_{12} \parallel T_{22}}{\vdash T_{11} \rightarrow T_{12} \parallel T_{21} \rightarrow T_{22}} \text{S\_FUN} \quad \frac{\vdash T_1 \parallel T_2 \quad \vdash_m T_1 \quad \vdash_m T_2}{\vdash_m \bullet \parallel T_1 \Rightarrow T_2} \text{A\_NONE}
\end{array}$$

Expression typing

$$\begin{array}{c}
\boxed{\Gamma \vdash_m e : T} \\
\\
\frac{\vdash_m \Gamma \quad x:T \in \Gamma}{\Gamma \vdash_m x : T} \text{T\_VAR} \quad \frac{\vdash_m T_1 \quad \Gamma, x:T_1 \vdash_m e_{12} : T_2}{\Gamma \vdash_m \lambda x:T_1. e_{12} : T_1 \rightarrow T_2} \text{T\_ABS} \quad \frac{\vdash_m \Gamma \quad \vdash_m T}{\Gamma \vdash_m \uparrow l : T} \text{T\_BLAME} \\
\\
\frac{\vdash_m \Gamma \quad \vdash_m \{x:B \mid e\} \quad \text{ty}(k) = B \quad e[k/x] \rightarrow_m^* \text{true}}{\Gamma \vdash_m k : \{x:B \mid e\}} \text{T\_CONST} \quad \frac{\text{ty}(op) = T_1 \rightarrow \dots \rightarrow T_n \rightarrow T \quad \Gamma \vdash_m e_i : T_i}{\Gamma \vdash_m op(e_1, \dots, e_n) : T} \text{T\_OP} \\
\\
\frac{\Gamma \vdash_m e_1 : (T_1 \rightarrow T_2) \quad \Gamma \vdash_m e_2 : T_1}{\Gamma \vdash_m e_1 e_2 : T_2} \text{T\_APP} \quad \frac{\vdash_m a \parallel T_1 \Rightarrow T_2 \quad \Gamma \vdash_m e : T_1}{\Gamma \vdash_m \langle T_1 \xrightarrow{a} T_2 \rangle^l e : T_2} \text{T\_CAST} \\
\\
\frac{\vdash_m \Gamma \quad \vdash_m \{x:B \mid e_1\} \quad \text{ty}(k) = B \quad \emptyset \vdash_m e_2 : \{x:\text{Bool} \mid \text{true}\} \quad e_1[k/x] \rightarrow_m^* e_2}{\Gamma \vdash_m \langle \{x:B \mid e_1\}, e_2, k \rangle^l : \{x:B \mid e_1\}} \text{T\_CHECK}
\end{array}$$

Figure 4: Universal typing rules of  $\lambda_H$

and codomains are well formed in mode  $m$ . (Unlike many recent formulations, our functions are not dependent—we leave dependency as future work.) Type compatibility  $\vdash T_1 \parallel T_2$  identifies types which can be cast to each other: the types must have the same “skeleton”. It is reasonable to try to cast a non-zero integer  $\{x:\text{Int} \mid x \neq 0\}$  to a positive integer  $\{x:\text{Int} \mid x > 0\}$ , but it is senseless to cast it to a boolean  $\{x:\text{Bool} \mid \text{true}\}$  or to a function type  $T_1 \rightarrow T_2$ . Every cast must be between compatible types; at their core,  $\lambda_H$  programs are well typed simply typed lambda calculus programs. Type compatibility is reflexive, symmetric, and transitive; i.e., it is an equivalence relation.

Our family of calculi use different annotations. All source programs (defined below) begin without annotations—we write the empty annotation  $\bullet$ . The universal annotation well formedness rule just defers to type compatibility (A\_NONE); it is an invariant that  $\vdash_m a \parallel T_1 \Rightarrow T_2$  implies  $\vdash T_1 \parallel T_2$ .

As for term typing, the T\_VAR, T\_ABS, T\_OP, and T\_APP rules are entirely conventional. T\_BLAME types blame at any (well formed) type. A constant  $k$  can be typed by T\_CONST at any type  $\{x:B \mid e\}$  in mode  $m$  if: (a)  $k$  is a  $B$ , i.e.,  $\text{ty}(k) = B$ ; (b) the type in question is well formed in  $m$ ; and (c), if  $e[k/x] \rightarrow_m^* \text{true}$ . As an immediate consequence, we can derive the following rule typing constants at their raw type, since  $\text{true} \rightarrow_m^* \text{true}$  in all modes and raw types are well formed in all modes (WF\_BASE):

$$\frac{\vdash_m \Gamma \quad \text{ty}(k) = B}{\Gamma \vdash_m k : \{x:B \mid \text{true}\}}$$

This approach to typing constants in a manifest calculus is novel: it offers a great deal of latitude with typing, while avoiding the subtyping of some formulations Greenberg et al. [2012], Flanagan [2006], Knowles and Flanagan [2010], Knowles et al. [2006] and the extra rule of others Belo et al. [2011]. We assume that  $\text{ty}(k) = \text{Bool}$  iff  $k \in \{\text{true}, \text{false}\}$ .

We require in T\_OP that  $\text{ty}(op)$  only produces well formed first-order types, i.e., types of the form  $\vdash_m \{x:B_1 \mid e_1\} \rightarrow \dots \rightarrow \{x:B_n \mid e_n\}$ . We require that the type is consistent with the operation’s denotation:  $\llbracket op \rrbracket(k_1, \dots, k_n)$  is defined iff  $e_i[k_i/x] \rightarrow_m^* \text{true}$  for all  $m$ . For this evaluation to hold for every system we consider, the types assigned to operations can’t involve casts that both (a) stack and (b) can fail—because forgetful  $\lambda_H$  may skip them, leading to different typings. We believe this is not so stringent a requirement: the types for operations ought to be simple, e.g.  $\text{ty}(\text{div}) = \{x:\text{Real} \mid \text{true}\} \rightarrow \{y:\text{Real} \mid y \neq 0\} \rightarrow \{z:\text{Real} \mid \text{true}\}$ , and stacked casts only arise in stack-free terms

due to function proxies. In general, it is interesting to ask what refinement types to assign to constants, as careless assignments can lead to circular checking (e.g., if division has a codomain cast checking its work with multiplication and vice versa).

The typing rule for casts, T\_CAST, relies on the annotation well formedness rule:  $\langle T_1 \xRightarrow{a} T_2 \rangle^l e$  is well formed in mode  $m$  when  $\vdash_m a \parallel T_1 \Rightarrow T_2$  and  $e$  is a  $T_1$ . Allowing any cast between compatible base types is conservative: a cast from  $\{x:\text{Int} \mid x > 0\}$  to  $\{x:\text{Int} \mid x \leq 0\}$  always fails. Earlier work has used SMT solvers to try to statically reject certain casts and eliminate those that are guaranteed to succeed Flanagan [2006], Knowles et al. [2006], Bierman et al. [2010]; we omit these checks, as we view them as secondary—a static analysis offering bug-finding and optimization, and not the essence of the system.

The final rule, T\_CHECK, is used for checking active checks, which should only occur at runtime. In fact, they should only ever be applied to closed terms; the rule allows for any well formed context as a technical device for weakening (Lemma A.1).

Active checks  $\langle \{x:B \mid e_1\}, e_2, k \rangle^l$  arise as the result of casts between refined base types, as in the following classic  $\lambda_H$  evaluation of a successful cast:

$$\begin{aligned} \langle \{x:B \mid e\} \xRightarrow{\bullet} \{x:B \mid e'\} \rangle^l k &\longrightarrow_C \langle \{x:B \mid e'\}, e'[k/x], k \rangle^l \\ &\longrightarrow_C^* \langle \{x:B \mid e'\}, \text{true}, k \rangle^l \\ &\longrightarrow_C k \end{aligned}$$

If we are going to prove type soundness via syntactic methods Wright and Felleisen [1994], we must have enough information to type  $k$  at  $\{x:B \mid e'\}$ . For this reason, T\_CHECK requires that  $e_1[k/x] \longrightarrow_m^* e_2$ ; this way, we know that  $e'[k/x] \longrightarrow_m^* \text{true}$  at the end of the previous derivation, which is enough to apply T\_CONST. The other premises of T\_CHECK ensure that the types all match up: that the target refinement type is well formed, that  $k$  as the base type in question, and that  $e_2$ , the current state of the active check, is also well formed.

To truly say that our languages share a syntax and a type system, we highlight a subset of type derivations as *source program* type derivations. We show that source programs well typed in one mode are well typed in the all modes (Appendix A).

**3.1 Definition [Source program]:** A source program type derivation obeys the following rules:

- T\_CONST only ever assigns the type  $\{x:\text{ty}(k) \mid \text{true}\}$ . Variations in each mode’s evaluation aren’t reflected in the (source program) type system. (We could soundly relax this requirement to allow  $\{x:\text{ty}(k) \mid e\}$  such that  $e[k/x] \longrightarrow_m^* \text{true}$  for any mode  $m$ .)
- Casts have empty annotations  $a = \bullet$ . Casts also have blame labels, and not empty blame (also written  $\bullet$ ).
- T\_CHECK, T\_STACK (Section 6), and T\_BLAKE are not used—these are for runtime only.

Note that source programs don’t use any of the typing rules that defer to the evaluation relation (T\_CHECK and T\_STACK), so we can maintain a clear phase distinction between type checking programs and running them.

### 3.4 Metatheory

One distinct advantage of having a single syntax with parameterized semantics is that some of the metatheory can be done once for all modes. Each mode proves its own canonical forms lemma—since each mode has a unique notion of value—and its own progress and preservation lemmas for syntactic type soundness Wright and Felleisen [1994]. But other standard metatheoretical machinery—weakening, substitution, and regularity—can be proved for all modes at once (see Section A.1). To wit, we prove syntactic type soundness in Appendix A.2 for classic  $\lambda_H$  in just three mode-specific lemmas: canonical forms, progress, and preservation. In every theorem statement, we include a reference to the lemma number where it is proved in the appendix. In PDF versions, this reference is hyperlinked.

### 3.5 Overview

In the rest of this paper, we give the semantics for three space-efficient modes for  $\lambda_H$ , relating the languages’ behavior on source programs (Definition 3.1). The forgetful mode is space efficient without annotations, converging to a value more often than classic  $\lambda_H$  ( $m = F$ ; Section 4). The heedful mode is space efficient and uses type sets to converge to a value exactly when classic  $\lambda_H$  does; it may blame different labels, though ( $m = H$ ; Section 5). The eidetic mode is space efficient and uses coercions to track pending checks; it behaves exactly like classic  $\lambda_H$  ( $m = E$ ; Section 6). We show that source programs that are well typed in one mode are well typed in all of them (Lemmas A.20, A.27, and A.35). We relate these space-efficient modes back to classic  $\lambda_H$  in Section 7.

$$\begin{array}{c}
\frac{}{\text{val}_m k} \quad \text{V\_CONST} \\
\\
\frac{}{\text{val}_m \lambda x:T. e} \quad \text{V\_ABS} \\
\\
\frac{}{\text{val}_F \langle T_{11} \rightarrow T_{12} \xRightarrow{\emptyset} T_{21} \rightarrow T_{22} \rangle^l \lambda x:T. e} \quad \text{V\_PROXYF} \\
\\
\text{merge}_F(T_1, \bullet, T_2, \bullet, T_3) = \bullet
\end{array}$$

Figure 5: Operational semantics of forgetful  $\lambda_H$ 

One may wonder why we even bother to mention forgetful and heedful  $\lambda_H$ , if eidetic  $\lambda_H$  is soundly space efficient with respect to classic  $\lambda_H$ . These two ‘intermediate’ modes are interesting as an exploration of the design space—but also in their own right.

Forgetful  $\lambda_H$  takes a radical approach that involves skipping checks—its soundness is rather surprising and offers insights into the semantics of contracts. Contracts have been used for more than avoiding wrongness, though: they have been used in PLT Racket for abstraction and information hiding PLT [a,b]. Forgetful  $\lambda_H$  can’t use contracts for information hiding. Suppose we implement user records as functions from strings to strings. We would like to pass a user record to an untrusted component, hiding some fields but not others. We can achieve this by specifying a white- or blacklist in a contract, e.g.,  $\{f:\text{String} \mid f \neq \text{“password”}\} \rightarrow \{v:\text{String} \mid \text{true}\}$ . Wrapping a function in this contract introduces a function proxy... which can be overwritten by `E_CASTMERGE`! To really get information hiding, the programmer must explicitly  $\eta$ -expand the function proxy, writing  $(\lambda f:\{f:\text{String} \mid f \neq \text{“password”}\}. \dots)$ . Forgetful  $\lambda_H$ ’s contracts can’t enforce abstractions.<sup>1</sup>

While Siek and Wadler [2010] uses the lattice of type precision in their threesomes without blame, our heedful  $\lambda_H$  uses the powerset lattice of types. Just as Siek and Wadler use labeled types and meet-like composition for threesomes *with* blame, we may be able to derive something similar for heedful and eidetic  $\lambda_H$ : in a (non-commutative) skew lattice, heedful uses a potentially re-ordering conjunction while eidetic preserves order. A lattice-theoretic account of casts, coercions, and blame may be possible.

## 4 Forgetful space efficiency

In forgetful  $\lambda_H$ , we offer a simple solution to space-inefficient casts: just forget about them. Function proxies only ever wrap lambda abstractions; trying to cast a function proxy simply throws away the inner proxy. Just the same, when accumulating casts on the stack, we throw away all but the last cast. Readers may wonder: how can this ever be sound? Several factors work together to make forgetful  $\lambda_H$  a sound calculus. In short, the key ingredients are call-by-value evaluation and the observation that type safety only talks about reduction to values in this setting.

In this section, our mode  $m = F$ : our evaluation relation is  $\rightarrow_F$  and we use typing judgments of the form, e.g.  $\Gamma \vdash_F e : T$ . Forgetful  $\lambda_H$  is the simplest of the space-efficient calculi: it just uses the standard typing rules from Figure 4 and the space-efficient reduction rules from Figure 2. We give the new operational definitions for  $m = F$  in Figure 5: a new value rule and the definition of the `merge` operator. First, `V_PROXYF` says that function proxies in forgetful  $\lambda_H$  are only values when the proxied value is a lambda (and not another function proxy). Limiting the number of function proxies is critical for establishing space bounds, as we do in Section 8. Forgetful casts don’t use annotations, so they just use `A_NONE`. The forgetful merge operator just *forgets* the intermediate type  $T_2$ .

We demonstrate this semantics on the example from Section 3.1 in Figure 6. We first step by merging casts, forgetting the intermediate type. Then contract checking proceeds as normal for the target type; since  $-1$  is non-zero, the check succeeds and returns its scrutinee by `E_CHECKOK`.

The type soundness property typically has two parts: (a) well typed programs don’t go ‘wrong’ (for us, getting stuck), and (b) well typed programs reduce to programs that are well typed at the same type. How could a forgetful  $\lambda_H$  program go wrong, violating property (a)? The general “skeletal” structure of types means we never have to worry about errors caught by simple type systems, such as trying to apply a non-function. Our semantics can get stuck by trying to apply an operator to an input that isn’t in its domain, e.g., trying to divide by zero. To guarantee that we avoid stuck operators,  $\lambda_H$  generally relies on subject reduction, property (b). Operators are assigned types

<sup>1</sup>This observation is due to Sam Tobin-Hochstadt.

$$\begin{aligned}
e &= \langle \{x:\text{Int} \mid x \bmod 2 = 0\} \dot{\Rightarrow} \{x:\text{Int} \mid x \neq 0\} \rangle^{l_3} \\
&\quad (\langle \{x:\text{Int} \mid x \geq 0\} \dot{\Rightarrow} \{x:\text{Int} \mid x \bmod 2 = 0\} \rangle^{l_2} \\
&\quad (\langle \{x:\text{Int} \mid \text{true}\} \dot{\Rightarrow} \{x:\text{Int} \mid x \geq 0\} \rangle^{l_1} -1)) \\
&\quad \text{(E\_CASTMERGE)} \\
&\longrightarrow_F \langle \{x:\text{Int} \mid x \geq 0\} \dot{\Rightarrow} \{x:\text{Int} \mid x \neq 0\} \rangle^{l_3} \\
&\quad (\langle \{x:\text{Int} \mid \text{true}\} \dot{\Rightarrow} \{x:\text{Int} \mid x \geq 0\} \rangle^{l_1} -1) \\
&\quad \text{(E\_CASTMERGE)} \\
&\longrightarrow_F \langle \{x:\text{Int} \mid \text{true}\} \dot{\Rightarrow} \{x:\text{Int} \mid x \neq 0\} \rangle^{l_3} -1 \\
&\quad \text{(E\_CHECKNONE)} \\
&\longrightarrow_F \langle \{x:\text{Int} \mid x \neq 0\}, -1 \neq 0, -1 \rangle^{l_3} \\
&\quad \text{(E\_CHECKINNER/E\_OP)} \\
&\longrightarrow_F \langle \{x:\text{Int} \mid x \neq 0\}, \text{true}, -1 \rangle^{l_3} \quad \text{(E\_CHECKOK)} \\
&\longrightarrow_F -1
\end{aligned}$$

Figure 6: Example of forgetful  $\lambda_H$

that avoid stuckness, i.e.,  $\text{ty}(op)$  and  $\llbracket op \rrbracket$  agree. Some earlier systems have done this Flanagan [2006], Knowles and Flanagan [2010] while others haven't Greenberg et al. [2012], Belo et al. [2011]. We view it as a critical component of contract calculi. So for, say, integer division,  $\text{ty}(\text{div}) = \{x:\text{Int} \mid \text{true}\} \rightarrow \{y:\text{Int} \mid y \neq 0\} \rightarrow \{z:\text{Int} \mid \text{true}\}$ . To actually use  $\text{div}$  in a program, the second argument must be typed as a non-zero integer—by a non-source typing with  $\text{T\_CONST}$  directly (see Definition 3.1) or by casting ( $\text{T\_CAST}$ ). It may seem dangerous: casts protect operators from improper values, preventing stuckness; forgetful  $\lambda_H$  eliminates some casts. But consider the cast eliminated by  $\text{E\_CASTMERGE}$ :

$$\langle T_2 \dot{\Rightarrow} T_3 \rangle^l (\langle T_1 \dot{\Rightarrow} T_2 \rangle^{l'} e) \longrightarrow_F \langle T_1 \dot{\Rightarrow} T_3 \rangle^l e$$

While the program tried to cast  $e$  to a  $T_2$ , it immediately cast it back out—no operation relies on  $e$  being a  $T_2$ . Skipping the check doesn't risk stuckness. Since  $\lambda_H$  is call-by-value, we can use the same reasoning to allow functions to assume that their inputs inhabit their types—a critical property for programmer reasoning.

Forgetful  $\lambda_H$  enjoys soundness via a standard syntactic proof of progress and preservation, reusing the theorems from Section A.1. What's more, source programs are well typed in classic  $\lambda_H$  iff they are well typed in forgetful  $\lambda_H$ : both languages can run the same terms. Proofs are in Appendix A.3.

## 5 Heedful space efficiency

Heedful  $\lambda_H$  ( $m = H$ ) takes the cast merging strategy from forgetful  $\lambda_H$ , but uses *type sets* on casts and function proxies to avoid dropping casts. Space efficiency for heedful  $\lambda_H$  rests on the use of sets: classic  $\lambda_H$  allows for arbitrary lists of function proxies and casts on the stack to accumulate. Restricting this accumulation to a set gives us a straightforward bound on the amount of accumulation: a program of fixed size can only have so many types at each size. We discuss this idea further in Section 8.

We extend the typing rules and operational semantics in Figure 7. Up until this point, we haven't used annotations. Heedful  $\lambda_H$  collects type sets as casts merge to record the types that must be checked. The  $\text{A\_TYPESET}$  annotation well formedness rule extends the premises of  $\text{A\_NONE}$  with the requirement that if  $\vdash_H \mathcal{S} \parallel T_1 \Rightarrow T_2$ , then all the types in  $\mathcal{S}$  are well formed and compatible with  $T_1$  and  $T_2$ . Type set compatibility is stable under removing elements from the set  $\mathcal{S}$ , and it is symmetric and transitive with respect to its type indices (since compatibility itself is symmetric and transitive).

One might expect the types in type sets to carry blame labels—might we then be able to have *sound* space efficiency? It turns out that just having labels in the sets isn't enough—we actually need to keep track of the ordering of checks. Eidetic  $\lambda_H$  (Section 6) does exactly this tracking. Consider this calculus a warmup.

Heedful  $\lambda_H$  adds some evaluation rules to the universal ones found in Figure 2. First  $\text{E\_TYPESET}$  takes a source program cast without an annotation and annotates it with an empty set.  $\text{E\_CHECKEMPTY}$  is exactly like  $\text{E\_CHECKNONE}$ , though we separate the two to avoid conflating the empty annotation  $\bullet$  and the empty set  $\emptyset$ . In  $\text{E\_CHECKSET}$ , we use an essentially unspecified function `choose` to pick a type from a type set to check. Using the `choose` function is theoretically expedient, as it hides all of heedful  $\lambda_H$ 's nondeterminism. Nothing is inherently problematic with this nondeterminism, but putting it in the reduction relation itself complicates the proof of strong normalization that is necessary for the proof relating classic and heedful  $\lambda_H$  (Section 7).

Type set well formedness

$$\boxed{\vdash_m \mathcal{S} \parallel T_1 \Rightarrow T_2}$$

$$\frac{\begin{array}{c} \vdash T_1 \parallel T_2 \quad \vdash_H T_1 \quad \vdash_H T_2 \\ \forall T \in \mathcal{S}. \vdash_H T \quad \vdash T \parallel T_1 \end{array}}{\vdash_H \mathcal{S} \parallel T_1 \Rightarrow T_2} \quad \text{A\_TYPESET}$$

Values and operational semantics

$$\boxed{\text{val}_H e}$$

$$\boxed{e_1 \longrightarrow_H e_2}$$

$$\frac{}{\text{val}_H \langle T_{11} \rightarrow T_{12} \xRightarrow{\mathcal{S}} T_{21} \rightarrow T_{22} \rangle^l \lambda x: T. e} \quad \text{V\_PROXYH}$$

$$\frac{}{\langle T_1 \xRightarrow{\bullet} T_2 \rangle^l e \longrightarrow_H \langle T_1 \xRightarrow{\emptyset} T_2 \rangle^l e} \quad \text{E\_TYPESET}$$

$$\frac{}{\begin{array}{c} \langle \{x:B \mid e_1\} \xRightarrow{\emptyset} \{x:B \mid e_2\} \rangle^l k \longrightarrow_H \\ \langle \{x:B \mid e_2\}, e_2[k/x], k \rangle^l \end{array}} \quad \text{E\_CHECKEMPTY}$$

$$\frac{\text{choose}(\mathcal{S}) = \{x:B \mid e_2\}}{\begin{array}{c} \langle \{x:B \mid e_1\} \xRightarrow{\mathcal{S}} \{x:B \mid e_3\} \rangle^l k \longrightarrow_H \\ \langle \{x:B \mid e_2\} \xRightarrow{\mathcal{S} \setminus \{x:B \mid e_2\}} \{x:B \mid e_3\} \rangle^l \\ \langle \{x:B \mid e_2\}, e_2[k/x], k \rangle^l \end{array}} \quad \text{E\_CHECKSET}$$

$$\text{choose}(\mathcal{S}) \in \mathcal{S} \text{ when } \mathcal{S} \neq \emptyset$$

$$\text{merge}_H(T_1, \mathcal{S}_1, T_2, \mathcal{S}_2, T_3) = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \{T_2\}$$

$$\begin{array}{lcl} \text{dom}(\mathcal{S}) & = & \bigcup_{T \in \mathcal{S}} \text{dom}(T) \\ \text{cod}(\mathcal{S}) & = & \bigcup_{T \in \mathcal{S}} \text{cod}(T) \end{array}$$

Figure 7: Annotation typing and operational semantics of heedful  $\lambda_H$

$$\begin{aligned}
e &= \langle \{x:\text{Int} \mid x \bmod 2 = 0\} \xRightarrow{\bullet} \{x:\text{Int} \mid x \neq 0\} \rangle^{l_3} \\
&\quad (\langle \{x:\text{Int} \mid x \geq 0\} \xRightarrow{\bullet} \{x:\text{Int} \mid x \bmod 2 = 0\} \rangle^{l_2} \\
&\quad (\langle \{x:\text{Int} \mid \text{true}\} \xRightarrow{\bullet} \{x:\text{Int} \mid x \geq 0\} \rangle^{l_1} -1)) \\
&\quad (\text{E\_TYPESET}, \text{E\_CASTINNER}/\text{E\_TYPESET}) \\
\longrightarrow_{\text{H}}^* &\langle \{x:\text{Int} \mid x \bmod 2 = 0\} \xRightarrow{\emptyset} \{x:\text{Int} \mid x \neq 0\} \rangle^{l_3} \\
&\quad (\langle \{x:\text{Int} \mid x \geq 0\} \xRightarrow{\emptyset} \{x:\text{Int} \mid x \bmod 2 = 0\} \rangle^{l_2} \\
&\quad (\langle \{x:\text{Int} \mid \text{true}\} \xRightarrow{\bullet} \{x:\text{Int} \mid x \geq 0\} \rangle^{l_1} -1)) \\
&\quad (\text{E\_CASTMERGE}) \\
\longrightarrow_{\text{H}} &\langle \{x:\text{Int} \mid x \geq 0\} \xRightarrow{\{\{x:\text{Int} \mid x \bmod 2 = 0\}\}} \{x:\text{Int} \mid x \neq 0\} \rangle^{l_3} \\
&\quad (\langle \{x:\text{Int} \mid \text{true}\} \xRightarrow{\bullet} \{x:\text{Int} \mid x \geq 0\} \rangle^{l_1} -1) \\
&\quad (\text{E\_CASTINNER}/\text{E\_TYPESET}) \\
\longrightarrow_{\text{H}} &\langle \{x:\text{Int} \mid x \geq 0\} \xRightarrow{\{\{x:\text{Int} \mid x \bmod 2 = 0\}\}} \{x:\text{Int} \mid x \neq 0\} \rangle^{l_3} \\
&\quad (\langle \{x:\text{Int} \mid \text{true}\} \xRightarrow{\emptyset} \{x:\text{Int} \mid x \geq 0\} \rangle^{l_1} -1) \\
&\quad (\text{E\_CASTMERGE}) \\
\longrightarrow_{\text{H}} &\langle \{x:\text{Int} \mid \text{true}\} \xRightarrow{\mathcal{S}} \{x:\text{Int} \mid x \neq 0\} \rangle^{l_3} -1 \\
&\quad \text{where } \mathcal{S} = \{\{x:\text{Int} \mid x \bmod 2 = 0\}, \{x:\text{Int} \mid x \geq 0\}\} \\
&\quad (\text{E\_CHECKSET}, \text{choose}(\mathcal{S}) = \{x:\text{Int} \mid x \bmod 2 = 0\}) \\
\longrightarrow_{\text{H}} &\langle \{x:\text{Int} \mid x \bmod 2 = 0\} \xRightarrow{\mathcal{S}'} \{x:\text{Int} \mid x \neq 0\} \rangle^{l_3} \\
&\quad (\{x:\text{Int} \mid x \bmod 2 = 0\}, -1 \bmod 2 = 0, -1)^{l_3} \\
&\quad \text{where } \mathcal{S}' = \{\{x:\text{Int} \mid x \geq 0\}\} \\
&\quad (\text{E\_CASTINNER}/\text{E\_CHECKINNER}/\text{E\_OP}) \\
\longrightarrow_{\text{H}} &\langle \{x:\text{Int} \mid x \bmod 2 = 0\} \xRightarrow{\mathcal{S}'} \{x:\text{Int} \mid x \neq 0\} \rangle^{l_3} \\
&\quad (\{x:\text{Int} \mid x \bmod 2 = 0\}, 1 = 0, -1)^{l_3} \\
&\quad (\text{E\_CASTINNER}/\text{E\_CHECKINNER}/\text{E\_OP}) \\
\longrightarrow_{\text{H}} &\langle \{x:\text{Int} \mid x \bmod 2 = 0\} \xRightarrow{\mathcal{S}'} \{x:\text{Int} \mid x \neq 0\} \rangle^{l_3} \\
&\quad (\{x:\text{Int} \mid x \bmod 2 = 0\}, \text{false}, -1)^{l_3} \\
&\quad (\text{E\_CASTINNER}/\text{E\_CHECKFAIL}) \\
\longrightarrow_{\text{H}} &\langle \{x:\text{Int} \mid x \geq 0\} \xRightarrow{\emptyset} \{x:\text{Int} \mid x \neq 0\} \rangle^{l_3} \uparrow^{l_3} \\
&\quad (\text{E\_CASTRAISE}) \\
\longrightarrow_{\text{H}} &\uparrow^{l_3}
\end{aligned}$$

Figure 8: Example of heedful  $\lambda_{\text{H}}$

For function types, we define  $\text{dom}(\mathcal{S})$  and  $\text{cod}(\mathcal{S})$  by mapping the underlying function on types over the set. Note that this may shrink the size of the set  $\mathcal{S}$ , but never grow it—there can't be more unique (co)domain types in  $\mathcal{S}$  than there are types.

The merge operator, used in  $\text{E\_CASTMERGE}$ , merges two sets by unioning the two type sets with the intermediate type, i.e.:

$$\langle T_2 \xRightarrow{\mathcal{S}_2} T_3 \rangle^{l_2} (\langle T_1 \xRightarrow{\mathcal{S}_1} T_2 \rangle^{l_1} e) \longrightarrow_{\text{H}} \langle T_1 \xRightarrow{\mathcal{S}_1 \cup \mathcal{S}_2 \cup \{T_2\}} T_3 \rangle^{l_2} e$$

There are some subtle interactions here between the different annotations: we won't merge casts that haven't yet stepped by  $\text{E\_TYPESET}$  because  $\text{merge}_{\text{H}}(T_1, \bullet, T_2, \bullet, T_3)$  isn't defined.

We demonstrate the heedful semantics by returning to the example from Section 3.1 in Figure 8. To highlight the difference between classic and heedful  $\lambda_{\text{H}}$ , we select a **choose** function that has heedful check the refinements out of order, failing on the check for evenness rather than the check for positivity. The real source of difference, however, is that  $\text{E\_CASTMERGE}$  takes the second blame label of the two casts it merges. Taking the first wouldn't be right, either: suppose that the target type of the  $l_1$  cast wasn't  $\{x:\text{Int} \mid x \geq 0\}$ , but some other type that  $-1$  inhabits. Then classic  $\lambda_{\text{H}}$  would blame  $l_2$ , but heedful  $\lambda_{\text{H}}$  would have held onto  $l_1$ . The solution to this blame tracking problem is to hold onto blame labels in annotations—which, again, is exactly what we do in eidetic  $\lambda_{\text{H}}$ .

The syntactic proof of type soundness for heedful  $\lambda_{\text{H}}$  appears in an appendix in Appendix A.4. We also have “source typing” for heedful  $\lambda_{\text{H}}$ : source programs are well typed when  $m = \text{C}$  iff they are well typed when  $m = \text{H}$ . As a corollary, source programs are well typed in  $\text{F}$  if and only if they are well typed in  $\text{H}$ .

## 6 Eidetic space efficiency

Eidetic  $\lambda_H$  uses *coercions*, a more refined system of annotations than heedful  $\lambda_H$ 's type sets. Coercions do two critical things that type sets don't: they retain check order, and they track blame. Our coercions are ultimately inspired by those of Henglein [1994]; we discuss the relationship between our coercions and his in related work (Section 9). Recall the syntax of coercions from Figure 1:

$$\begin{aligned} c &::= r \mid c_1 \mapsto c_2 \\ r &::= \text{nil} \mid \{x:B \mid e\}^l, r \end{aligned}$$

Coercions come in two flavors: blame-annotated refinement lists  $r$ —zero or more refinement types, each annotated with a blame label—and function coercions  $c_1 \mapsto c_2$ . We write them as comma separated lists, omitting the empty refinement list  $\text{nil}$  when the refinement list is non-empty. We define the coercion well formedness rules, an additional typing rule, and reduction rules for eidetic  $\lambda_H$  in Figure 9. To ease the exposition, our explanation doesn't mirror the rule groupings in the figure.

As a general intuition, coercions are plans for checking: they contain precisely those types to be checked. Refinement lists are well formed for casts between  $\{x:B \mid e_1\}$  and  $\{x:B \mid e_2\}$  when: (a) every type in the list is a blame-annotated, well formed refinement of  $B$ , i.e., all the types are of the form  $\{x:B \mid e\}^l$  and are therefore similar to the indices; (b) there are no duplicated types in the list; and (c) the target type  $\{x:B \mid e_2\}$  is implied by some other type in the list. Note that the input type for all refinement lists can be any well formed refinement—this corresponds to the intuition that base types have no negative parts, i.e., casts between refinements ignore the type on the left. Finally, we simply write “no duplicates in  $r$ ”—it is an invariant during the evaluation of source programs. Function coercions, on the other hand, have a straightforward (contravariant) well formedness rule.

The  $\text{E\_COERCE}$  rule translates source-program casts to coercions:  $\text{coerce}(T_1, T_2, l)$  is a coercion representing exactly the checking done by the cast  $\langle T_1 \dot{\Rightarrow} T_2 \rangle^l$ . All of the refinement types in  $\text{coerce}(T_1, T_2, l)$  are annotated with the blame label  $l$ , since that's the label that would be blamed if the cast failed at that type. Since a coercion is a complete plan for checking, a coercion annotation obviates the need for type indices and blame labels. To this end,  $\text{E\_COERCE}$  drops the blame label from the cast, replacing it with an empty label. We keep the type indices so that we can reuse  $\text{E\_CASTMERGE}$  from the universal semantics, and also as a technical device in the preservation proof.

The actual checking of coercions rests on the treatment of refinement lists: function coercions are expanded as functions are applied by  $\text{E\_UNWRAP}$ , so they don't need much special treatment beyond a definition for  $\text{dom}$  and  $\text{cod}$ . Eidetic  $\lambda_H$  uses *coercion stacks*  $\langle \{x:B \mid e_1\}, s, r, k, e \rangle^\bullet$  to evaluate refinement lists. Coercion stacks are type checked by  $\text{T\_STACK}$  (in Figure 9). We explain the operational semantics before explaining the typing rule. Coercion stacks are runtime-only entities comprising five parts: a target type, a status, a pending refinement list, a constant scrutinee, and a checking term. We keep the target type of the coercion for preservation's sake. The status bit  $s$  is either  $\checkmark$  or  $?$ : when the status is  $\checkmark$ , we are currently checking or have already checked the target type  $\{x:B \mid e_1\}$ ; when it is  $?$ , we haven't. The pending refinement list  $r$  holds those checks not yet done. When  $s = ?$ , the target type is still in  $r$ . The scrutinee  $k$  is the constant we're checking; the checking term  $e$  is *either* the scrutinee  $k$  itself, or it is an active check on  $k$ .

The evaluation of a coercion stack proceeds as follows. First,  $\text{E\_COERCSTACK}$  starts a coercion stack when a cast between refinements meets a constant, recording the target type, setting the status to  $?$ , and setting the checking term to  $k$ . Then  $\text{E\_STACKPOP}$  starts an active check on the first type in the refinement list, using its blame label on the active check—possibly updating the status if the type being popped from the list is the target type. The active check runs by the congruence rule  $\text{E\_STACKINNER}$ , eventually returning  $k$  itself or blame. In the latter case,  $\text{E\_STACKRAISE}$  propagates the blame. If not, then the scrutinee is  $k$  once more and  $\text{E\_STACKPOP}$  can fire again. Eventually, the refinement list is exhausted, and  $\text{E\_STACKDONE}$  returns  $k$ .

Now we can explain  $\text{T\_STACK}$ 's many jobs. It must recapitulate  $\text{A\_REFINE}$ , but not exactly—since eventually the target type will be checked and no longer appear in  $r$ . The status  $s$  differentiates what our requirement is: when  $s = ?$ , the target type is in  $r$ . When  $s = \checkmark$ , we either know that  $k$  inhabits the target type or that we are currently checking the target type (i.e., an active check of the target type at some blame label reduces to our current checking term).

Finally, we must define a merge operator,  $\text{merge}_E$ . We define it in terms of the  $\triangleright$  operator, which is very nearly concatenation on refinement lists and a contravariant homomorphism on function coercions. It's not concatenation because it uses an implication predicate, the pre-order  $\supset$ , to eliminate duplicates (because  $\supset$  is reflexive) and hide subsumed types (because  $\supset$  is adequate). We read  $\{x:B \mid e_1\} \supset \{x:B \mid e_2\}$  as “ $\{x:B \mid e_1\}$  implies  $\{x:B \mid e_2\}$ ”. When eliminating types,  $\triangleright$  always chooses the leftmost blame label. Contravariance means that  $c_1 \triangleright c_2$  takes leftmost labels in positive positions and rightmost labels in negative ones. The parentheses in the definition of  $r \setminus T$  (read “ $r$

**Coercion implication predicate: axioms**

$$\boxed{\{x:B \mid e_1\} \supset \{x:B \mid e_2\}}$$

(**Reflexivity**) If  $\vdash_E \{x:B \mid e\}$  then  $\{x:B \mid e\} \supset \{x:B \mid e\}$ .

(**Transitivity**) If  $\{x:B \mid e_1\} \supset \{x:B \mid e_2\}$  and  $\{x:B \mid e_2\} \supset \{x:B \mid e_3\}$  then  $\{x:B \mid e_1\} \supset \{x:B \mid e_3\}$ .

(**Adequacy**) If  $\{x:B \mid e_1\} \supset \{x:B \mid e_2\}$  then  $\forall k \in \mathcal{K}_B. e_1[k/x] \rightarrow_E^* \text{true}$  implies  $e_2[k/x] \rightarrow_E^* \text{true}$ .

(**Decidability**) For all  $\vdash_E \{x:B \mid e_1\}$  and  $\vdash_E \{x:B \mid e_2\}$ , it is decidable whether  $\{x:B \mid e_1\} \supset \{x:B \mid e_2\}$ .

**Coercion well formedness and term typing**

$$\boxed{\vdash_m c \parallel T_1 \Rightarrow T_2}$$

$$\boxed{\Gamma \vdash_m e : T}$$

$$\frac{\begin{array}{l} \vdash_E \{x:B \mid e_1\} \quad \vdash_E \{x:B \mid e_2\} \\ \forall \{x:B \mid e\} \in r. \vdash_E \{x:B \mid e\} \quad \text{no duplicates in } r \\ \exists \{x:B \mid e\} \in r. \{x:B \mid e\} \supset \{x:B \mid e_2\} \end{array}}{\vdash_E r \parallel \{x:B \mid e_1\} \Rightarrow \{x:B \mid e_2\}}$$

A\_REFINE

$$\frac{\vdash_E c_1 \parallel T_{21} \Rightarrow T_{11} \quad \vdash_E c_2 \parallel T_{12} \Rightarrow T_{22}}{\vdash_E c_1 \mapsto c_2 \parallel (T_{11} \rightarrow T_{12}) \Rightarrow (T_{21} \rightarrow T_{22})}$$

A\_FUN

$$\frac{\begin{array}{l} \vdash_E \Gamma \quad \vdash_E \{x:B \mid e_1\} \quad \text{ty}(k) = B \quad \emptyset \vdash_E e_2 : \{x:B \mid e_3\} \quad \forall \{x:B \mid e\} \in r. \vdash_E \{x:B \mid e\} \\ s = \checkmark \text{ implies } e_1[k/x] \rightarrow_E^* \text{true} \vee (\exists \{x:B \mid e\}. \exists l. \{x:B \mid e\} \supset \{x:B \mid e_1\} \wedge \langle \{x:B \mid e\}, e[k/x], k \rangle^l \rightarrow_E^* e_2) \\ s = ? \text{ implies } (\exists \{x:B \mid e\} \in r. \{x:B \mid e\} \supset \{x:B \mid e_1\}) \end{array}}{\Gamma \vdash_E \langle \{x:B \mid e_1\}, s, r, k, e_2 \rangle^\bullet : \{x:B \mid e_2\}}$$

T\_STACK

**Values and operational semantics**

$$\boxed{\text{val}_E e}$$

$$\boxed{e_1 \rightarrow_E e_2}$$

$$\frac{}{\text{val}_E \langle T_{11} \rightarrow T_{12} \xrightarrow{c_1 \mapsto c_2} T_{21} \rightarrow T_{22} \rangle^\bullet \lambda x. T. e} \quad \text{V_PROXYE}$$

$$\frac{}{\langle T_1 \xrightarrow{\bullet} T_2 \rangle^l e \rightarrow_E \langle T_1 \xrightarrow{\text{coerce}(T_1, T_2, l)} T_2 \rangle^\bullet e} \quad \text{E_COERCE}$$

$$\frac{}{\langle \{x:B \mid e_1\} \xrightarrow{\tau} \{x:B \mid e_2\} \rangle^\bullet k \rightarrow_E \langle \{x:B \mid e_2\}, ?, r, k, k \rangle^\bullet} \quad \text{E_COERCESTACK}$$

$$\frac{}{\langle \{x:B \mid e\}, s, (\{x:B \mid e'\}^l, r), k, k \rangle^\bullet \rightarrow_E \langle \{x:B \mid e\}, s \vee (e = e'), r, k, \langle \{x:B \mid e'\}, e'[k/x], k \rangle^l \rangle^\bullet} \quad \text{E_STACKPOP}$$

$$\frac{e' \rightarrow_E e''}{\langle \{x:B \mid e\}, s, r, k, e' \rangle^\bullet \rightarrow_E \langle \{x:B \mid e\}, s, r, k, e'' \rangle^\bullet} \quad \text{E_STACKINNER} \quad \frac{}{\langle \{x:B \mid e\}, s, r, k, \uparrow l' \rangle^\bullet \rightarrow_E \uparrow l'} \quad \text{E_STACKRAISE}$$

$$\frac{}{\langle \{x:B \mid e\}, \checkmark, \text{nil}, k, k \rangle^\bullet \rightarrow_E k} \quad \text{E_STACKDONE}$$

**Cast translation and coercion operations**

$$\begin{array}{llll} \text{merge}_E(T_1, c_1, T_2, c_2, T_3) & = & c_1 \triangleright c_2 & \text{coerce}(\{x:B \mid e_1\}, \{x:B \mid e_2\}, l) = \{x:B \mid e_2\}^l \\ \text{dom}(c_1 \mapsto c_2) & = & c_1 & \text{coerce}(T_{11} \rightarrow T_{12}, T_{21} \rightarrow T_{22}, l) = \text{coerce}(T_{21}, T_{11}, l) \mapsto \text{coerce}(T_{12}, T_{22}, l) \\ \text{cod}(c_1 \mapsto c_2) & = & c_2 & \end{array}$$

$$\begin{array}{llll} \{x:B \mid e\}^l \triangleright \text{nil} & = & \{x:B \mid e\}^l & \text{nil} \triangleright r_2 = r_2 \\ \{x:B \mid e\}^l \triangleright r & = & \{x:B \mid e\}^l, (r \setminus \{x:B \mid e\}) & (\{x:B \mid e\}^l, r_1) \triangleright r_2 = \{x:B \mid e\}^l \triangleright (r_1 \triangleright r_2) \\ & & & (c_{11} \mapsto c_{12}) \triangleright (c_{21} \mapsto c_{22}) = (c_{21} \triangleright c_{11}) \mapsto (c_{12} \triangleright c_{22}) \end{array}$$

$$\begin{array}{ll} \text{nil} \setminus \{x:B \mid e\} & = \text{nil} \\ (\{x:B \mid e_1\}^l, r) \setminus \{x:B \mid e\} & = \begin{cases} r \setminus \{x:B \mid e\} & \{x:B \mid e\} \supset \{x:B \mid e_1\} \\ \{x:B \mid e_1\}^l, (r \setminus \{x:B \mid e\}) & \{x:B \mid e\} \not\supset \{x:B \mid e_1\} \end{cases} \end{array}$$

$$\begin{array}{ll} \checkmark \vee (e_1 = e_2) & = \checkmark \\ ? \vee (e_1 = e_2) & = \begin{cases} \checkmark & e_1 = e_2 \\ ? & \text{otherwise} \end{cases} \end{array}$$

Figure 9: Typing rules and operational semantics for eidetic  $\lambda_H$



dropping  $T''$ ) isn't a typo: the `coerce` metafunction and  $\triangleright$  operator work together to make sure that the refinement lists are correctly ordered. As we show below, 'correctly ordered' means the positive parts take older labels and negative parts take newer ones. As for heedful  $\lambda_H$ , `E_CASTMERGE` is slightly subtle—we never merge casts with  $\bullet$  as an annotation because such merges aren't defined.

In Figure 9, we only give the axioms for  $\triangleright$ : it must be an adequate, decidable pre-order. Syntactic type equality is the simplest implementation of the  $\triangleright$  predicate, but the reflexive transitive closure of any adequate decidable relation would work.

By way of example, consider a cast from  $T_1 = \{x:\text{Int} \mid x \geq 0\} \rightarrow \{x:\text{Int} \mid x \geq 0\}$  to  $T_2 = \{x:\text{Int} \mid \text{true}\} \rightarrow \{x:\text{Int} \mid x > 0\}$ . For brevity, we refer to the domains as  $T_{i1}$  and the codomains as  $T_{i2}$ . We find that  $(\langle T_1 \xrightarrow{\bullet} T_2 \rangle^l v_1) v_2$  steps in classic  $\lambda_H$  to:

$$\begin{aligned} & \langle \{x:\text{Int} \mid x \geq 0\} \xrightarrow{\bullet} \{x:\text{Int} \mid x > 0\} \rangle^l \\ & (v_1 (\langle \{x:\text{Int} \mid \text{true}\} \xrightarrow{\bullet} \{x:\text{Int} \mid x \geq 0\} \rangle^l v_2)) \end{aligned}$$

Note that  $T_1$ 's domain is checked but its codomain isn't; the reverse is true for  $T_2$ . When looking at a cast, we can read off which refinements are checked by looking at the positive parts of the target type and the negative parts of the source type. The relationship between casts and polarity is not a new one Findler [2006], Gronski and Flanagan [2007], Herman et al. [2007], Wadler and Findler [2009], Greenberg [2013]. Unlike casts, coercions directly express the sequence of checks to be performed. Consider the coercion generated from the cast above, recalling that  $T_{i1}$  and  $T_{i2}$  are the domains and codomains of  $T_1$  and  $T_2$ :

$$\begin{aligned} & \langle \langle T_1 \xrightarrow{\bullet} T_2 \rangle^l v_1 \rangle v_2 \\ \rightarrow_E & \langle \langle T_1 \xrightarrow{\bullet} T_2 \rangle^{\bullet} v_1 \rangle v_2 \\ & \text{where } c = \{x:\text{Int} \mid x \geq 0\}^l \mapsto \{x:\text{Int} \mid x > 0\}^l \\ \rightarrow_E & \langle \langle T_{11} \rightarrow T_{12} \xrightarrow{\bullet} T_{21} \rightarrow T_{22} \rangle^{\bullet} v_1 \rangle v_2 \\ \rightarrow_E & \langle T_{12}^{\{x:\text{Int} \mid x > 0\}^l} T_{22} \rangle^{\bullet} (v_1 (\langle T_{21}^{\{x:\text{Int} \mid x \geq 0\}^l} T_{11} \rangle^{\bullet} v_2)) \end{aligned}$$

In this example, there is only a single blame label,  $l$ . Tracking blame labels is critical for exactly matching classic  $\lambda_H$ 's behavior. The example below relies on reflexivity of  $\triangleright$ . To see why, we return to our example from before in Figure 10. Throughout the merging, each refinement type retains its own original blame label, allowing eidetic  $\lambda_H$  to behave just like classic  $\lambda_H$ .

We offer a final pair of examples, showing how coercions with redundant types are merged. The intuition here is that positive positions are checked covariantly—oldest (innermost) cast first—while negative positions are checked contravariantly—newest (outermost) cast first. Consider the classic  $\lambda_H$  term:

$$\begin{aligned} T_1 &= \{x:\text{Int} \mid e_{11}\} \rightarrow \{x:\text{Int} \mid e_{21}\} \\ T_2 &= \{x:\text{Int} \mid e_{12}\} \rightarrow \{x:\text{Int} \mid e_{22}\} \\ T_3 &= \{x:\text{Int} \mid e_{13}\} \rightarrow \{x:\text{Int} \mid e_{22}\} \\ e &= \langle T_2 \xrightarrow{\bullet} T_3 \rangle^{l_2} (\langle T_1 \xrightarrow{\bullet} T_2 \rangle^{l_1} v) \end{aligned}$$

Note that the casts run inside-out, from old to new in the positive position, but they run from the outside-in, new to old, in the negative position.

$$\begin{aligned} e \ v' & \rightarrow_C \langle \{x:\text{Int} \mid e_{22}\} \xrightarrow{\bullet} \{x:\text{Int} \mid e_{22}\} \rangle^{l_2} \\ & (\langle \{x:\text{Int} \mid e_{21}\} \xrightarrow{\bullet} \{x:\text{Int} \mid e_{22}\} \rangle^{l_1} \\ & (v (\langle \{x:\text{Int} \mid e_{12}\} \xrightarrow{\bullet} \{x:\text{Int} \mid e_{12}\} \rangle^{l_1} \\ & (\langle \{x:\text{Int} \mid e_{13}\} \xrightarrow{\bullet} \{x:\text{Int} \mid e_{12}\} \rangle^{l_2} v'))))) \end{aligned}$$

The key observation for eliminating redundant checks is that only the check run first can fail—there's no point in checking a predicate contract twice on the same value. So eidetic  $\lambda_H$  merges like so:

$$\begin{aligned} e & \rightarrow_E^* \langle T_2^{\{x:\text{Int} \mid e_{12}\}^{l_2}} \xrightarrow{\bullet} \{x:\text{Int} \mid e_{22}\}^{l_2} T_3 \rangle^{\bullet} \\ & (\langle T_1^{\{x:\text{Int} \mid e_{11}\}^{l_1}} \xrightarrow{\bullet} \{x:\text{Int} \mid e_{22}\}^{l_1} T_2 \rangle^{\bullet} v) \\ & \rightarrow_E \langle T_1 \xrightarrow{\bullet} T_3 \rangle^{\bullet} v \end{aligned}$$

where

$$\begin{aligned} c &= (\{x:\text{Int} \mid e_{12}\}^{l_2} \triangleright \{x:\text{Int} \mid e_{11}\}^{l_1}) \mapsto \\ & (\{x:\text{Int} \mid e_{22}\}^{l_1} \triangleright \{x:\text{Int} \mid e_{22}\}^{l_2}) \\ &= \{x:\text{Int} \mid e_{12}\}^{l_2}, \{x:\text{Int} \mid e_{11}\}^{l_1} \mapsto \{x:\text{Int} \mid e_{22}\}^{l_1} \end{aligned}$$

$$\begin{aligned}
e &= \langle \{x:\text{Int} \mid x \bmod 2 = 0\} \xRightarrow{\bullet} \{x:\text{Int} \mid x \neq 0\} \rangle^{l_3} \\
&\quad (\langle \{x:\text{Int} \mid x \geq 0\} \xRightarrow{\bullet} \{x:\text{Int} \mid x \bmod 2 = 0\} \rangle^{l_2} \\
&\quad (\langle \{x:\text{Int} \mid \text{true}\} \xRightarrow{\bullet} \{x:\text{Int} \mid x \geq 0\} \rangle^{l_1} -1)) \\
&\quad \quad \quad (\text{E\_COERCE}) \\
&\longrightarrow_E \langle \{x:\text{Int} \mid x \bmod 2 = 0\} \xRightarrow{\{x:\text{Int} \mid x \neq 0\}^{l_3}} \{x:\text{Int} \mid x \neq 0\} \rangle^{l_3} \\
&\quad (\langle \{x:\text{Int} \mid x \geq 0\} \xRightarrow{\bullet} \{x:\text{Int} \mid x \bmod 2 = 0\} \rangle^{l_2} \\
&\quad (\langle \{x:\text{Int} \mid \text{true}\} \xRightarrow{\bullet} \{x:\text{Int} \mid x \geq 0\} \rangle^{l_1} -1)) \\
&\quad \quad \quad (\text{E\_CASTINNER/E\_COERCE}) \\
&\longrightarrow_E \langle \{x:\text{Int} \mid x \bmod 2 = 0\} \xRightarrow{\{x:\text{Int} \mid x \neq 0\}^{l_3}} \{x:\text{Int} \mid x \neq 0\} \rangle^{l_3} \\
&\quad (\langle \{x:\text{Int} \mid x \geq 0\} \xRightarrow{\{x:\text{Int} \mid x \bmod 2 = 0\}^{l_2}} \{x:\text{Int} \mid x \bmod 2 = 0\} \rangle^{l_2} \\
&\quad (\langle \{x:\text{Int} \mid \text{true}\} \xRightarrow{\bullet} \{x:\text{Int} \mid x \geq 0\} \rangle^{l_1} -1)) \\
&\quad \quad \quad (\text{E\_CASTMERGE}) \\
&\longrightarrow_E \langle \{x:\text{Int} \mid x \geq 0\} \xRightarrow{r'} \{x:\text{Int} \mid x \neq 0\} \rangle^{l_3} \\
&\quad (\langle \{x:\text{Int} \mid \text{true}\} \xRightarrow{\bullet} \{x:\text{Int} \mid x \geq 0\} \rangle^{l_1} -1) \\
&\quad \text{where } r' = \{x:\text{Int} \mid x \bmod 2 = 0\}^{l_2}, \{x:\text{Int} \mid x \neq 0\}^{l_3} \\
&\quad \quad \quad (\text{E\_CASTINNER/E\_COERCE}) \\
&\longrightarrow_E \langle \{x:\text{Int} \mid x \geq 0\} \xRightarrow{r'} \{x:\text{Int} \mid x \neq 0\} \rangle^{l_3} \\
&\quad (\langle \{x:\text{Int} \mid \text{true}\} \xRightarrow{\{x:\text{Int} \mid x \geq 0\}^{l_1}} \{x:\text{Int} \mid x \geq 0\} \rangle^{l_1} -1) \\
&\quad \quad \quad (\text{E\_CASTMERGE}) \\
&\longrightarrow_E \langle \{x:\text{Int} \mid \text{true}\} \xRightarrow{r} \{x:\text{Int} \mid x \neq 0\} \rangle^{l_3} -1 \\
&\quad \text{where } r = \{x:\text{Int} \mid x \geq 0\}^{l_1}, r' \\
&\quad \quad \quad (\text{E\_COERCESTACK}) \\
&\longrightarrow_E \langle \{x:\text{Int} \mid x \neq 0\}, ?, r, -1, -1 \rangle^\bullet \\
&\quad \quad \quad (\text{E\_STACKPOP}) \\
&\longrightarrow_E \langle \{x:\text{Int} \mid x \neq 0\}, ?, r', -1, \\
&\quad \quad \langle \{x:\text{Int} \mid x \geq 0\}, -1 \geq 0, -1 \rangle^{l_1} \rangle^\bullet \\
&\longrightarrow_E^* \uparrow_{l_1}
\end{aligned}$$

Figure 10: Example of eidetic  $\lambda_H$

The coercion merge operator eliminates the redundant codomain check, choosing to keep the one with blame label  $l_1$ . Choosing  $l_1$  makes sense here because the codomain is a positive position and  $l_1$  is the older, innermost cast. We construct a similar example for merges in negative positions.

$$\begin{aligned}
T_1 &= \{x:\text{Int} \mid e_{11}\} \rightarrow \{x:\text{Int} \mid e_{21}\} \\
T_2' &= \{x:\text{Int} \mid e_{11}\} \rightarrow \{x:\text{Int} \mid e_{22}\} \\
T_3' &= \{x:\text{Int} \mid e_{13}\} \rightarrow \{x:\text{Int} \mid e_{23}\} \\
e' &= \langle T_2' \xRightarrow{\bullet} T_3' \rangle^{l_2} (\langle T_1 \xRightarrow{\bullet} T_2' \rangle^{l_1} v)
\end{aligned}$$

Again, the unfolding runs the positive parts inside-out and the negative parts outside-in when applied to a value  $v'$ :

$$\begin{aligned}
&\langle \{x:\text{Int} \mid e_{22}\} \xRightarrow{\bullet} \{x:\text{Int} \mid e_{23}\} \rangle^{l_2} \\
&\quad (\langle \{x:\text{Int} \mid e_{21}\} \xRightarrow{\bullet} \{x:\text{Int} \mid e_{22}\} \rangle^{l_1} \\
&\quad \quad (v (\langle \{x:\text{Int} \mid e_{11}\} \xRightarrow{\bullet} \{x:\text{Int} \mid e_{11}\} \rangle^{l_1} \\
&\quad \quad \quad (\langle \{x:\text{Int} \mid e_{13}\} \xRightarrow{\bullet} \{x:\text{Int} \mid e_{11}\} \rangle^{l_2} v')))))
\end{aligned}$$

Running the example in eidetic  $\lambda_H$ , we reduce the redundant checks in the domain:

$$\begin{aligned}
e' &\longrightarrow_E^* \langle T_2' \xRightarrow{\{x:\text{Int} \mid e_{11}\}^{l_2} \mapsto \{x:\text{Int} \mid e_{23}\}^{l_2}} T_3' \rangle^\bullet \\
&\quad (\langle T_1 \xRightarrow{\{x:\text{Int} \mid e_{11}\}^{l_1} \mapsto \{x:\text{Int} \mid e_{22}\}^{l_1}} T_2' \rangle^\bullet v) \\
&\longrightarrow_E \langle T_1 \xRightarrow{c} T_3' \rangle^\bullet v \\
&\text{where} \\
c &= (\{x:\text{Int} \mid e_{11}\}^{l_2} \triangleright \{x:\text{Int} \mid e_{11}\}^{l_1}) \mapsto \\
&\quad (\{x:\text{Int} \mid e_{22}\}^{l_1} \triangleright \{x:\text{Int} \mid e_{23}\}^{l_2}) \\
&= \{x:\text{Int} \mid e_{12}\}^{l_2} \mapsto \{x:\text{Int} \mid e_{22}\}^{l_1}, \{x:\text{Int} \mid e_{23}\}^{l_2}
\end{aligned}$$

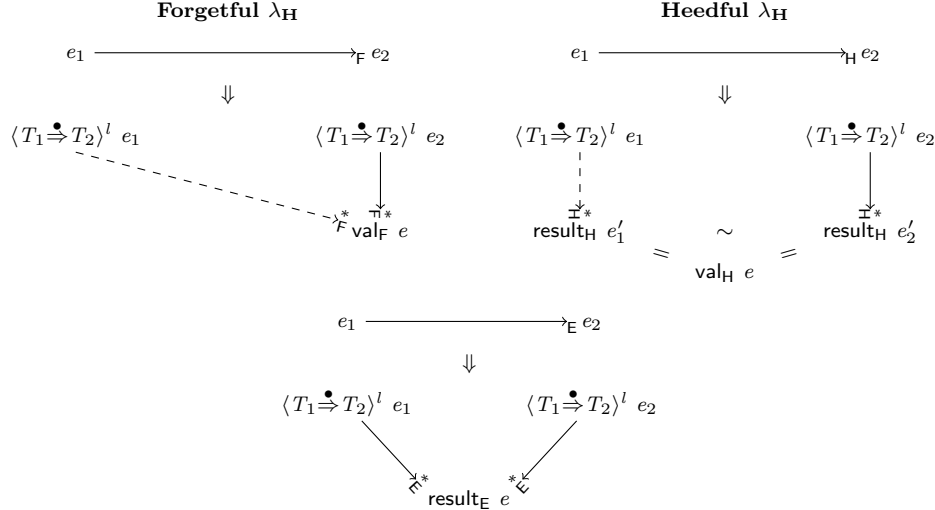


Figure 11: Cast congruence lemmas as commutative diagrams

Following the outside-in rule for negative positions, we keep the blame label  $l_2$  from the newer, outermost cast.

As we did for the other calculi, we present the routine syntactic proof of type soundness in (Appendix A.5).

Like forgetful and heedful  $\lambda_H$  before, eidetic  $\lambda_H$  shares source programs (Definition 3.1) with classic  $\lambda_H$ . With this final lemma, we know that all modes share the same well typed source programs.

## 7 Soundness for space efficiency

We want space efficiency to be *sound*: it would be space efficient to never check anything. Classic  $\lambda_H$  is normative: the more a mode behaves like classic  $\lambda_H$ , the “sounder” it is.

A single property summarizes how a space-efficient calculus behaves with respect to classic  $\lambda_H$ : cast congruence. In classic  $\lambda_H$ , if  $e_1 \rightarrow_C e_2$  then  $\langle T_1 \Rightarrow T_2 \rangle^l e_1$  and  $\langle T_1 \Rightarrow T_2 \rangle^l e_2$  behave identically. This cast congruence principle is easy to see, because  $E\_CASTINNERC$  applies freely. In the space-efficient modes, however,  $E\_CASTINNER$  can only apply when  $E\_CASTMERGE$  doesn’t. Merged casts may not behave the same as running the two casts separately. We summarize the results in commutative diagrams in Figure 11. Forgetful  $\lambda_H$  has the property that if the unmerged casts reduce to a value, then so do the merged ones. But the merged casts may reduce to a value when the unmerged ones reduce to blame, because forgetful merging skips checks. Heedful  $\lambda_H$  has a stronger property: the merged and unmerged casts *coterminate* at results, if the merged term reduces to blame or a value, so does the unmerged term. If they both go to values, they go to the exact same value; but if they both go to blame, they may blame different labels. This is a direct result of  $E\_CASTMERGE$  saving only one label from casts. Finally, eidetic  $\lambda_H$  has a property as strong as heedful  $\lambda_H$ : the merged and unmerged casts coterminate exactly.

It is particularly nice that the key property for relating modes can be proved entirely within each mode, i.e., the cast congruence lemma for forgetful  $\lambda_H$  is proved *independently* of classic  $\lambda_H$ .

The proofs are in Appendix B, but there are two points worth observing here. First, we need strong normalization to prove cast congruence for heedful  $\lambda_H$ : if we reorder checks, we need to know that reordering checks doesn’t change the observable behavior. Second, both heedful and eidetic  $\lambda_H$  eliminate redundant checks when merging casts, the former by using sets and the latter by means of the  $\triangleright$  operator and the reflexivity of  $\triangleright$ . These two calculi require proofs that checking is idempotent: checking a property once is as good as checking it twice. Naturally, this property only holds without state.

We summarize the relationships between each mode in Table 1. Our proofs relating classic  $\lambda_H$  and the space-efficient modes are by (mode-indexed)logical relations, found in Figure 12. The relation is modal: in  $e_1 \sim_m e_2 : T$ , the term  $e_1$  is a classic  $\lambda_H$  term, while  $e_2$  and  $T$  are in mode  $m$ . Each mode’s logical relation matches its cast congruence lemma: the forgetful logical relation allows more blame on the classic side (not unlike the asymmetric logical relations of Greenberg et al. [2012]); the heedful logical relation is blame-inexact, allowing classic and heedful  $\lambda_H$  to raise different labels; the eidetic logical relation is exact. The proofs can be found in Appendix B. They follow a fairly standard pattern in each mode  $m$ : we show that applying C-casts and  $m$ -casts between similar and

Mode	Reduction behavior
Classic ( $m = C$ )	Normative
Forgetful ( $m = F$ )	$\rightarrow_C^* \text{val} \Rightarrow \rightarrow_F^* \text{val}$ (Lemma B.6)
Heedful ( $m = H$ )	$\rightarrow_C^* \text{result} \Leftrightarrow \rightarrow_H^* \text{result}$ (Lemma B.20)
Eidetic ( $m = E$ )	$\rightarrow_C^* \text{result} = \rightarrow_E^* \text{result}$ (Lemma B.25)

Table 1: Soundness results for  $\lambda_H$

Value rules

$$\boxed{e_1 \sim_m e_2 : T}$$

$$\begin{aligned} k \sim_m k : \{x:B \mid e\} &\iff \text{ty}(k) = B \wedge e[k/x] \rightarrow_m^* \text{true} \\ e_{11} \sim_m e_{21} : T_1 \rightarrow T_2 &\iff \text{val}_C e_{11} \wedge \text{val}_m e_{12} \wedge \\ &\quad \forall e_{12} \sim_m e_{22} : T_1. e_{11} e_{12} \simeq_m e_{21} e_{22} : T_2 \end{aligned}$$

Term rules

$$\boxed{e_1 \Downarrow_m e_2 : T}$$

$$\boxed{e_1 \simeq_m e_2 : T}$$

$$\begin{aligned} e_1 \Downarrow_m e_2 : T &\iff e_1 \rightarrow_C^* e'_1 \wedge \text{val}_C e'_1 \wedge \\ &\quad e_2 \rightarrow_m^* e'_2 \wedge \text{val}_m e'_2 \wedge \\ &\quad e'_1 \sim_m e'_2 : T \end{aligned}$$

$$\begin{aligned} e_1 \simeq_F e_2 : T &\iff e_1 \rightarrow_C^* \uparrow l \vee e_1 \Downarrow_F e_2 : T \\ e_1 \simeq_H e_2 : T &\iff (e_1 \rightarrow_C^* \uparrow l \wedge e_2 \rightarrow_H^* \uparrow l') \vee e_1 \Downarrow_H e_2 : T \\ e_1 \simeq_E e_2 : T &\iff (e_1 \rightarrow_C^* \uparrow l \wedge e_2 \rightarrow_E^* \uparrow l) \vee e_1 \Downarrow_E e_2 : T \end{aligned}$$

Type rules

$$\boxed{T_1 \sim_m T_2}$$

$$\begin{aligned} \{x:B \mid e_1\} \sim_m \{x:B \mid e_2\} &\iff \\ &\quad \forall e'_1 \sim_m e'_2 : \{x:B \mid \text{true}\}. \\ &\quad e_1[e'_1/x] \simeq_m e_2[e'_2/x] : \{x:\text{Bool} \mid \text{true}\} \\ T_{11} \rightarrow T_{12} \sim_m T_{21} \rightarrow T_{22} &\iff T_{11} \sim_m T_{21} \wedge T_{12} \sim_m T_{22} \end{aligned}$$

Closing substitutions and open terms

$$\boxed{\Gamma \models_m \delta}$$

$$\boxed{\Gamma \vdash e_1 \simeq_m e_2 : T}$$

$$\begin{aligned} \Gamma \models_m \delta &\iff \forall x \in \text{dom}(\Gamma). \delta_1(x) \sim_m \delta_2(x) : \Gamma(x) \\ \Gamma \vdash e_1 \simeq_m e_2 : T &\iff \forall \Gamma \models_m \delta. \delta_1(e_1) \simeq_m \delta_2(e_2) : T \end{aligned}$$

Figure 12: Modal logical relations relating classic  $\lambda_H$  to space-efficient modes

Mode	Cast size	Pending casts
Classic ( $m = C$ )	$2W_h + L$	$\infty$
Forgetful ( $m = F$ )	$2W_h + L$	$ e $
Heedful ( $m = H$ )	$2W_h + 2^{W_h} + L$	$ e $
Eidetic ( $m = E$ )	$s2^{L+W_B}$	$ e $

Table 2: Space efficiency of  $\lambda_H$

related types to related values yields related values (i.e., casts are applicative); we then show that well typed C-source programs are related to  $m$ -source programs. As far as alternative techniques go, an induction over evaluation derivations wouldn’t give us enough information about evaluations that return lambda abstractions. Other contextual equivalence techniques (e.g., bisimulation) would probably work, too.

Our equivalence results for forgetful and heedful  $\lambda_H$  are subtle: they would break down if we had effects other than blame. Forgetful  $\lambda_H$  changes which contracts are checked, and so which code is run. Heedful  $\lambda_H$  can reorder when code is run. Well typed  $\lambda_H$  programs in this paper are strongly normalizing. If we allowed nontermination, for example, then we could construct source programs that diverge in classic  $\lambda_H$  and converge in forgetful  $\lambda_H$ , or source programs that diverge in one of classic and heedful  $\lambda_H$  and converge in the other. Similarly, if blame were a *catchable* exception, we would have no relation for these two modes at all: since they can raise different blame labels, different exception handlers could have entirely different behavior. Eidetic  $\lambda_H$  doesn’t reorder checks, though, so its result is more durable. As long as checks are pure—they return the same result every time—eidetic and classic  $\lambda_H$  coincide.

Why bother proving strong results for forgetful and heedful if they only adhere in such a restricted setting? First, we wish to explore the design space. Moreover, forgetful and heedful offer insights into the semantics and structure of casts. Second, we want to show soundness of space efficiency in isolation. Implementations always differ from the theory. Once this soundness has been established, whether an inefficient classic implementation would differ for a given program is less relevant when the reference implementation has a heedful semantics. Analogously, languages with first-class stack traces make tail-call optimization observable, but this change in semantics is typically considered worthwhile—space efficiency is more important.

## 8 Bounds for space efficiency

We have claimed that forgetful, heedful, and eidetic  $\lambda_H$  are space efficient: what do we mean? What sort of space efficiency have we achieved in our various calculi? We summarize the results in Table 2; proofs are in Appendix C. From a high level, there are only a finite number of types that appear in our programs, and this set of types can only reduce as the program runs. We can effectively code each type in the program as an integer, allowing us to efficiently run the  $\supset$  predicate.

Suppose that a type of height  $h$  can be represented in  $W_h$  bits and a label in  $L$  bits. (Type heights are defined in Figure 14 in Appendix C.) Casts in classic and forgetful  $\lambda_H$  each take up  $2W_h + L$  bits: two types and a blame label. Casts in heedful  $\lambda_H$  take up more space— $2W_h + 2^{W_h} + L$  bits—because they need to keep track of the type set. Coercions in eidetic  $\lambda_H$  have a different form: the only types recorded are those of height 1, i.e., refinements of base types. Pessimistically, each of these may appear at every position in a function coercion  $c_1 \mapsto c_2$ . We use  $s$  to indicate the “size” of a function type, i.e., the number of positions it has. As a first pass, a set of refinements and blame labels take up  $2^{L+W_1}$  space. But in fact these coercions must all be between refinements of *the same base type*, leading to  $2^{L+W_B}$  space per coercion, where  $W_B$  is the highest number of refinements of any single base type. We now have our worst-case space complexity:  $s2^{L+W_B}$ . A more precise bound might track which refinements appear in which parts of a function type, but in the worst case—each refinement appears in every position—it degenerates to the bound we give here. Classic  $\lambda_H$  can have an infinite number of “pending casts”—casts and function proxies—in a program. Forgetful, heedful and eidetic  $\lambda_H$  can have no more than one pending cast per term node. Abstractions are limited to a single function proxy, and E\_CASTMERGE merges adjacent pending casts.

The text of a program  $e$  is finite, so the set of types appearing in the program,  $\mathbf{types}(e)$ , is also finite. Since reduction doesn’t introduce types, we can bound the number of types in a program (and therefore the sizes of casts). We can therefore fix a numerical coding for types at runtime, where we can encode a type in  $W = \log_2(|\mathbf{types}(e)|)$  bits. In a given cast,  $W$  over-approximates how many types can appear: the source, target, and annotation must all be compatible, which means they must also be of the same height. We can therefore represent the types in casts with fewer bits:  $W_h = \log_2(|\{T \mid T \in \mathbf{types}(e) \wedge \mathbf{height}(T) = h\}|)$ . In the worst case, we revert to the original bound: all types in the program are of height 1. Even so, there are never casts between different base types  $B$  and  $B'$ , so

$W_B = \max_B \log_2(|\{x:B \mid e\} \in \mathbf{types}(e)|)$ . Eidetic  $\lambda_H$ 's coercions never hold types greater than height 1. The types on its casts are erasable once the coercions are generated, because coercions drive the checking.

The bounds we find here are *galactic*. Having established that contracts are theoretically space efficient, making an implementation practically space efficient is a different endeavor, involving careful choices of representations and calling conventions. We have shown that sound space efficiency is possible. It is future work to produce a feasible implementation.

Eidetic  $\lambda_H$ 's space bounds rely only on the reflexivity of the  $\supset$  predicate, since we leave it abstract. We have identified one situation where the relation allows us to find better space bounds: mutual implication.

If  $\{x:B \mid e_1\} \supset \{x:B \mid e_2\}$  and  $\{x:B \mid e_2\} \supset \{x:B \mid e_1\}$ , then these two types are equivalent, and only one ever need be checked. Which to check could be determined by a compiler with a suitably clever cost model. Note that our proofs don't quite bear out this optimization, though it is intuitively correct to us. By default, our  $\triangleright$  operator will take whichever of  $\{x:B \mid e_1\}$  and  $\{x:B \mid e_2\}$  was meant to be checked first. Adapting the  $\triangleright$  operator to always choose one based on some preference relation would not be particularly hard, and we believe the proofs adapt easily.

Other analyses of the relation seem promising at first, but in fact do not allow more compact representations. Suppose we have a program where  $\{x:B \mid e_1\} \supset \{x:B \mid e_2\}$  and  $B$  is our worst case type. That is,  $W_B = 2$ , because there are 2 different refinements of  $B$  and fewer refinements of other base types. The worst-case representation for a refinement list is 2 bits, with bit  $b_i$  indicating whether  $e_i$  is present in the list. Can we do any better than 2 bits, since  $e_1$  can stand in for  $e_2$ ? Could we represent the two types as just 1 bit? We cannot when (a) there are constants that pass one type but not the other and (b) when refinement lists are in the reverse order of implication. Suppose there is some  $k$  such that  $e_2[k/x] \rightarrow_E^* \text{true}$  and  $e_1[k/x] \rightarrow_E^* \text{false}$ . Now we consider a concatenation of refinement lists in the reverse ordering:  $\{x:B \mid e_2\}^l \triangleright \{x:B \mid e_1\}^{l'}$ . We must retain both checks, since different failures lead to different blame. The  $k$  that passes  $e_2$  but not  $e_1$  should raise  $\uparrow l'$ , but other  $k'$  that fail for both types should raise  $\uparrow l$ . One bit isn't enough to capture the situation of having the coercion  $\{x:B \mid e_2\}^l, \{x:B \mid e_1\}^{l'}$ .

Finally, what is the right representation for a function? When calling a function, do we need to run coercions or not? Jeremy Siek suggested a "smart closure" which holds the logic for branching inside its own code; this may support better branch prediction than an indirect jump or branching at call sites.

## 9 Related work

Some earlier work uses first-class casts, whereas our casts are always applied to a term Belo et al. [2011], Knowles and Flanagan [2010]. It is of course possible to  $\eta$ -expand a cast with an abstraction, so no expressiveness is lost. Leaving casts fully applied saves us from the puzzling rules managing how casts work on other casts in space-efficient semantics, like:  $\langle T_{11} \rightarrow T_{12} \xrightarrow{\bullet} T_{21} \rightarrow T_{22} \rangle^l \langle T_{11} \xrightarrow{\bullet} T_{12} \rangle^{l'} \rightarrow_F \langle T_{21} \xrightarrow{\bullet} T_{22} \rangle^l$ .

Previous approaches to space-efficiency have focused on gradual typing Siek and Taha [2006], using coercions Henglein [1994], casts, casts annotated with intermediate types  $a/k/a$  *threesomes*, or some combination of all three Siek et al. [2009], Siek and Wadler [2010], Herman et al. [2010], Siek and Garcia [2012], Garcia [2013]. Recent work relates all three frameworks, making particular use of coercions Anonymous [2014]. Our type structure differs from that of gradual types, so our space bounds come in a somewhat novel form. Gradual types, without the more complicated checking that comes with predicate contracts, allow for simpler designs. Siek and Wadler [2010] can define a simple recursive operator on labeled types with a strong relationship to subtyping, the fundamental property of casts. We haven't been able to discover a connection in our setting. Instead, we ignore the type structure of functions and focus our attention on managing labels in lists of first-order predicate contracts. Gradual types occasionally have simpler proofs, too, e.g., by induction on evaluation Siek et al. [2009]; even when strong reasoning principles are needed, the presence of dynamic types leads them to use bisimulation Siek and Wadler [2010], Garcia [2013], Anonymous [2014]. We use logical relations because  $\lambda_H$ 's type structure is readily available, and because they allow us to easily reason about how checks evaluate.

Our coercions are inspired by Henglein's coercions for modeling injection to and projection from the dynamic type Henglein [1994]. Henglein's primitive coercions tag and untag values, while ours represent checks to be performed on base types; both our formulation and Henglein's use structural function coercions.

Greenberg [2013], the most closely related work, offers a coercion language combining the dynamic types of Henglein's original work with predicate contracts; his EFFICIENT language does not quite achieve "sound" space efficiency. Rather, it is forgetful, occasionally dropping casts. He omits blame, though he conjectures that blame for coercions reads left to right (as it does in Siek and Garcia [2012]); our eidetic  $\lambda_H$  verifies this conjecture. While Greenberg's languages offer dynamic, simple, and refined types, our types here are entirely refined. His coercions use Henglein's  $!$  and  $?$  syntax for injection and projection, while our coercions lack such a distinction. In our refinement

lists, each coercion simultaneously projects from one refinement type and injects into another (possibly producing blame). We reduce notation by omitting the interrobang ‘?’.

Dimoulas et al. [2013] introduce *option contracts*, which offer a programmatic way of turning off contract checking, as well as a controlled way to “pass the buck”, handing off contracts from component to component. Option contracts address time efficiency, not space efficiency. Findler et al. [2008] studied space and time efficiency for datatype contracts, as did Koukoutos and Kuncak [2014].

PLT Racket contracts have a mild form of space efficiency: the `tail-marks-match?` predicate<sup>2</sup> checks for exact duplicate contracts at tail positions. The redundancy it detects seems to rely on pointer equality. Since PLT Racket contracts are (a) module-oriented “macro” contracts, and (b) first class, this optimization is somewhat unpredictable—and limited compared with our heedful and eidetic calculi.

```
#lang racket
```

```
(define (count-em-integer? x)
  (printf "checking ~s\n" x)
  (integer? x))

;; yes:
(letrec ([f
  (contract (-> any/c count-em-integer?)
    (lambda (x)
      (printf "x: ~s\n" x)
      (if (zero? x) x (f (- x 1))))
    'pos
    'neg)])
  (f 3))

;; no:
(letrec ([f
  (contract (-> any/c count-em-integer?)
    (contract (-> any/c count-em-integer?)
      (lambda (x)
        (printf "x: ~s\n" x)
        (if (zero? x) x (f (- x 1))))
      'pos
      'neg)
    'pos
    'neg)])
  (f 3))
```

## 10 Conclusion and future work

Semantics-preserving space efficiency for manifest contracts is possible—leaving the admissibility of state as the final barrier to practical utility. We established that eidetic  $\lambda_H$  behaves exactly like its classic counterpart without compromising space usage. Forgetful  $\lambda_H$  is an interesting middle ground: if contracts exist to make partial operations safe (and not abstraction or information hiding), forgetfulness may be a good strategy.

We believe it would be easy to design a latent version of eidetic  $\lambda_H$ , following the translations in Greenberg et al. [2010].

In our simple (i.e., not dependent) case, our refinement types close over a single variable of base type. Space efficiency for a dependent calculus remains open. The first step towards dependent types would be extending  $\supset$  with a context (and a source of closing substitutions, a serious issue Belo et al. [2011]). In a dependent setting the definition of what it means to compare closures isn’t at all clear. Closures’ environments may contain functions, and closures over extensionally equivalent functions may not be intensionally equal. A more nominal approach to contract comparison may resolve some of the issues here. Comparisons might be more straightforward when contracts are

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<sup>2</sup>From `racket/collects/racket/contract/private/arrow.rkt`.

explicitly declared and referenced by name. Similarly, a dependent  $\supset$  predicate might be more easily defined over some explicit structured family of types, like a lattice. Findler et al. [2008] has made some progress in this direction.

Finally, a host of practical issues remain. Beyond representation choices, having expensive checks makes it important to predict *when* checks happen. The  $\supset$  predicate compares closures and will surely have delicate interactions with optimizations.

## Acknowledgments

Comments from Rajeev Alur, Ron Garcia, Fritz Henglein, Greg Morrisett, Stephanie Weirich, Phil Wadler, and Steve Zdancewic improved a previous version of this work done at the University of Pennsylvania. Some comments from Benjamin Pierce led me to realize that a cast formulation was straightforward. Discussions with Atsushi Igarashi, Robby Findler, and Sam Tobin-Hochstadt greatly improved the quality of the exposition. Phil Wadler encouraged me to return to coercions to understand the eidetic formulation. The POPL reviewers had many excellent suggestions, and Robby Findler helped once more with angelic guidance.

This work was supported in part by the NSF under grants TC 0915671 and SHF 1016937 and by the DARPA CRASH program through the United States Air Force Research Laboratory (AFRL) under Contract No. FA8650-10-C-7090. The views expressed are the author's and do not reflect the official policy or position of the Department of Defense or the U.S. Government.

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## A Proofs of type soundness

This appendix includes the proofs of type soundness for all four modes of  $\lambda_H$ ; we first prove some universally applicable metatheoretical properties.

### A.1 Generic metatheory

**A.1 Lemma [Weakening]:** If  $\Gamma_1, \Gamma_2 \vdash_m e : T$  and  $\vdash_m T'$  and  $x$  is fresh, then  $\vdash_m \Gamma_1, x:T', \Gamma_2$  and  $\Gamma_1, x:T', \Gamma_2 \vdash_m e : T$ .

**Proof:** The context well formedness is by induction on  $\Gamma_2$ .

(WF\_EMPTY) By WF\_EXTEND.

(WF\_EXTEND) By WF\_EXTEND and the IH.

By induction on  $e$ , leaving  $\Gamma_2$  generalized.

( $e = y$ ) If  $y:T \in \Gamma_1, \Gamma_2$ , then  $y:T \in \Gamma_1, x:T', \Gamma_2$ .

( $e = k$ ) By T\_CONST and the first part of the IH.

( $e = \lambda y:T_1. e$ ) By T\_ABS and the IH on  $e$ , using  $\Gamma_2, y:T_1$ .

( $e = \langle T_1 \dot{\Rightarrow} T_2 \rangle^l e'$ ) By T\_CAST and the IHs.

( $e = \langle T_1 \dot{\Rightarrow} T_2 \rangle^\bullet e'$ ) By T\_COERCE and the IHs.

( $e = \uparrow l$ ) By T\_BLAKE and the IH.

( $e = e_1 e_2$ ) By T\_APP and the IHs.

( $e = op(e_1, \dots, e_n)$ ) By T\_OP and the IHs.

( $e = \langle \{x:B \mid e_1\}, e_2, k \rangle^l$ ) By T\_CHECK and the IH, observing that  $e_2$  and  $k$  are closed.  $\square$

**A.2 Lemma [Substitution]:** If  $\Gamma_1, x:T', \Gamma_2 \vdash_m e : T$  and  $\emptyset \vdash_m e' : T'$ , then  $\Gamma_1, \Gamma_2 \vdash_m e[e'/x] : T$  and  $\vdash_m \Gamma_1, \Gamma_2$ .

**Proof:** By induction on the typing derivation.

(T\_VAR) By the assumption of typing and weakening, Lemma A.1.

(T\_CONST) By T\_CONST and the IH (for  $\vdash_m \Gamma_1, \Gamma_2$ ).

(T\_ABS) By T\_ABS and the IH, renaming the variable.

(T\_OP) By T\_OP and the IHs.

(T\_APP) By T\_APP and the IHs.

(T\_CAST) By T\_CAST and the IHs, observing that all types are closed, though the term may not be.

(T\_COERCE) By T\_COERCE and the IHs, observing that all types are closed, though the term may not be.

(T\_BLAKE) By T\_BLAKE and the IH (for  $\vdash_m \Gamma_1, \Gamma_2$ ).

(T\_CHECK) By T\_CHECK and the IH (for  $\vdash_m \Gamma_1, \Gamma_2$ ). Note that the subterms are all closed, so the substitution actually has no effect other than strengthening the context.  $\square$

**A.3 Lemma [Coercion typing regularity]:** If  $\vdash_E c \parallel T_1 \Rightarrow T_2$  then  $\vdash_E T_1$  and  $\vdash_E T_2$ .

**Proof:** By induction on the coercion well formedness derivation.

(A\_REFINE) By assumption.

(A\_FUN) By WF\_FUN on the IHs.  $\square$

**A.4 Lemma [Regularity]:** If  $\Gamma \vdash_m e : T$ , then  $\vdash_m \Gamma$  and  $\vdash_m T$ .

**Proof:** By induction on the typing relation.

(T\_VAR) We have  $\vdash_m \Gamma$  by assumption, and  $\vdash_m T$  by induction on the derivation of  $\vdash_m \Gamma$ .

(T\_CONST) We have  $\vdash_m \Gamma$  and  $\vdash_m T$  by assumption.

(T\_ABS) By the IH, we get  $\vdash_m \Gamma, x:T_1$  and  $\vdash_m T_2$ . By inversion we find  $\vdash_m \Gamma$ ; by WF\_FUN and the assumption that  $\vdash_m T_1$ , we find  $\vdash_m T_1 \rightarrow T_2$ .

(T\_OP) We have  $\vdash_m \Gamma$  and  $\vdash_m T$  by assumption.  
(T\_APP) By the IH,  $\vdash_m T_1 \rightarrow T_2$  and  $\vdash_m \Gamma$ . By inversion,  $\vdash_m T_2$ .  
(T\_CAST) By WF\_FUN and inversion of the type set well formedness;  $\vdash_m \Gamma$  is by the IH on  $\Gamma \vdash_m e : T_1$ .  
(T\_COERCE) It must be that  $m = E$ ; by coercion regularity (Lemma A.3).  
(T\_BLAZE) By assumption.  
(T\_CHECK) By assumption.

□

**A.5 Lemma [Similarity is reflexive]:** If  $\vdash T \parallel T$ .

**Proof:** By induction on  $T$ .

( $T = \{x:B \mid e\}$ ) By S\_REFINE.  
( $T = T_1 \rightarrow T_2$ ) By S\_FUN and the IHs.

□

**A.6 Lemma [Similarity is symmetric]:** If  $\vdash T_1 \parallel T_2$ , then  $\vdash T_2 \parallel T_1$ .

**Proof:** By induction on the similarity derivation.

(S\_REFINE) By S\_REFINE.  
(S\_FUN) By S\_FUN and the IHs.

□

**A.7 Lemma [Similarity is transitive]:** If  $\vdash T_1 \parallel T_2$  and  $\vdash T_2 \parallel T_3$ , then  $\vdash T_1 \parallel T_3$ .

**Proof:** By induction on the derivation of  $\vdash T_1 \parallel T_2$ .

(S\_REFINE) The other derivation must also be by S\_REFINE; by S\_REFINE.  
(S\_FUN) The other derivation must also be by S\_FUN; by S\_FUN and the IHs.

□

**A.8 Lemma [Well formed type sets have similar indices]:**

If  $\vdash_m \mathcal{S} \parallel T_1 \Rightarrow T_2$  then  $\vdash T_1 \parallel T_2$ .

**Proof:** Immediate, by inversion.

□

**A.9 Lemma [Type set well formedness is symmetric]:**  $\vdash_m a \parallel T_1 \Rightarrow T_2$  iff  $\vdash_m a \parallel T_2 \Rightarrow T_1$  for all  $m \neq E$ .

**Proof:** We immediately have  $\vdash_m T_1$  and  $\vdash_m T_2$ , and  $\vdash T_1 \parallel T_2$  iff  $\vdash T_2 \parallel T_1$  by Lemma A.6.

If  $m = C$  or  $m = F$ , then by A\_NONE and symmetry of similarity (Lemma A.6).

If  $m = H$ , then let  $T \in \mathcal{S}$  be given. The  $\vdash_H T$  premises hold immediately; we are then done by transitivity (Lemma A.7) and symmetry (Lemma A.6) of similarity ( $\vdash T \parallel T_1$  iff  $\vdash T \parallel T_2$  when  $\vdash T_1 \parallel T_2$ ). □

**A.10 Lemma [Type set well formedness is transitive]:** If  $\vdash T_1 \parallel T_2$  and  $\vdash_m a \parallel T_2 \Rightarrow T_3$  and  $\vdash_m T_1$  and  $m \neq E$  then  $\vdash_m a \parallel T_1 \Rightarrow T_3$ .

**Proof:** We immediately have  $\vdash_m T_1$  and  $\vdash_m T_3$ ; we have  $\vdash T_1 \parallel T_3$  by transitivity of similarity (Lemma A.7).

If  $m = C$  or  $m = F$ , we are done immediately by A\_NONE.

If, on the other hand,  $m = H$ , let  $T \in \mathcal{S}$  be given. We know that  $\vdash_H T$  and  $\vdash T \parallel T_2$ ; by symmetry (Lemma A.6) and transitivity (Lemma A.7) of similarity, we are done by A\_TYPESET. □

**A.11 Lemma [Reducing type sets]:** If  $\vdash_m \mathcal{S} \parallel T_1 \Rightarrow T_3$  then  $\vdash_m (\mathcal{S} \setminus T_2) \parallel T_1 \Rightarrow T_3$ .

**Proof:** All of the  $T \in \mathcal{S}$  remain well formed and similar to  $T_1$  and  $T_3$ , as do the well formedness and similarity relations for  $T_1$  and  $T_3$ . □

## A.2 Classic type soundness

**A.12 Lemma [Classic determinism]:** If  $e \rightarrow_C e_1$  and  $e \rightarrow_C e_2$  then  $e_1 = e_2$ .

**Proof:** By induction on the first evaluation derivation.  $\square$

**A.13 Lemma [Classic canonical forms]:** If  $\emptyset \vdash_C e : T$  and  $\text{val}_C e$  then:

- If  $T = \{x:B \mid e'\}$ , then  $e = k$  and  $\text{ty}(k) = B$  and  $e'[e/x] \rightarrow_C^* \text{true}$ .
- If  $T = T_1 \rightarrow T_2$ , then either  $e = \lambda x:T. e'$  or  $e = \langle T_{11} \rightarrow T_{12} \xrightarrow{\bullet} T_{21} \rightarrow T_{22} \rangle^l e'$ .

**Proof:** By inversion of  $\text{val}_C e$  and inspection of the typing rules.  $V\_CONST/T\_CONST$  are the only rules that types values at base types;  $V\_ABS/T\_ABS$  and  $V\_PROXY/T\_CAST$  are the only rules that type values at function types.  $\square$

**A.14 Lemma [Classic progress]:** If  $\emptyset \vdash_C e : T$ , then either:

1.  $\text{result}_C e$ , i.e.,  $e = \uparrow l$  or  $\text{val}_C e$ ; or
2. there exists an  $e'$  such that  $e \rightarrow_C e'$ .

**Proof:** By induction on the typing derivation.

(T\_VAR) A contradiction— $x$  isn't well typed in the empty environment.

(T\_CONST)  $e = k$  is a result by  $V\_CONST$  and  $R\_VAL$ .

(T\_ABS)  $e = \lambda x:T. e'$  is a result by  $V\_ABS$  and  $R\_VAL$ .

(T\_OP) We know that  $\text{ty}(op)$  is a first order type  $\{x:B_1 \mid e'_1\} \rightarrow \dots \rightarrow \{x:B_n \mid e'_n\} \rightarrow T$ . From left to right, we apply the IH on  $e_i$ . If  $e_i$  is a result, there are two cases: either  $e_i = \uparrow l$ , and we step by  $E\_OPRAISE$ ; or  $e_i = k_i$ , since constants are the only values at base types by canonical forms (Lemma A.13), and we continue on to the next  $e_i$ . If any of the  $e_i$  step, we know that all of terms before them are values, so we can step by  $E\_OPINNER$ . If all of the  $e_i$  are constants  $k_i$ , then  $\emptyset \vdash_C k_i : \{x:B_i \mid e'_i\}$ , and so  $e'_i[k_i/x] \rightarrow_C^* \text{true}$ . Therefore  $\llbracket op \rrbracket(k_1, \dots, k_n)$  is defined, and we can step by  $E\_OP$ .

(T\_APP) By the IH on  $e_1$ , we know that  $e_1$  is blame, is a value, or steps to some  $e'_1$ . In the first case, we take a step by  $E\_APPRAISEL$ . In the latter, we take a step by  $E\_APPL$ .

If  $\text{val}_C e_1$ , then we can apply the IH on  $e_2$ , which is blame, is a value, or steps to some  $e'_2$ . The first and last cases are as before, using  $E\_APPRAISER$  and  $E\_APPR$ .

If  $\text{val}_C e_2$ , we must go by cases on the shape of  $e_1$ . Since  $\emptyset \vdash_C e_1 : T_1 \rightarrow T_2$ , by canonical forms (Lemma A.13) we know that  $e_1$  can only be an abstraction, a wrapped abstraction, or a cast.

( $e_1 = \lambda x:T_1. e'_1$ ) We step to  $e'_1[e_2/x]$  by  $E\_BETA$ .

( $e_1 = \langle T_{11} \rightarrow T_{12} \xrightarrow{\bullet} T_{21} \rightarrow T_{22} \rangle^l e'_1$ ) We step to  $\langle T_{12} \xrightarrow{\bullet} T_{22} \rangle^l (e'_1 (\langle T_{21} \xrightarrow{\bullet} T_{11} \rangle^l e_2))$  by  $E\_UNWRAP$ , noting that all of the type sets are empty.

(T\_CAST) If  $e' \rightarrow_C e''$ , then by  $E\_CASTINNERC$ . If  $e'$  is blame, then by  $E\_CASTRAISE$ . If  $\text{val}_C e'$ , then we invert  $\vdash T_1 \parallel T_2$ , finding that either:

( $T_i = \{x:B \mid e_i\}$ ) By canonical forms (Lemma A.13),  $e_2 = k$ . We can step to  $\langle \{x:B \mid e_2\}, e_2[k/x], k \rangle^l$  by  $E\_CHECKNONE$ .

( $T_i = T_{i1} \rightarrow T_{i2}$ ) This is a value,  $V\_PROXYC$ .

(T\_BLAZE)  $\uparrow l$  is a result by  $R\_BLAME$ .

(T\_CHECK) By the IH on  $e_2$ , we know that  $e_2$  is  $\uparrow l'$ , is a value by  $\text{val}_C e_2$ , or takes a step to some  $e'_2$ . In the first case, we step to  $\uparrow l'$  by  $E\_CHECKRAISE$ . In the last case, we step by  $E\_CHECKINNER$ . If  $\text{val}_C e_2$ , by canonical forms (Lemma A.13) we know that  $e_2$  is a  $k$  such that  $\text{ty}(k) = \text{Bool}$ , i.e.,  $e_2$  is either  $\text{true}$  or  $\text{false}$ . In the former case, we step to  $k$  by  $E\_CHECKOK$ ; in the latter case, we step to  $\uparrow l$  by  $E\_CHECKFAIL$ .  $\square$

**A.15 Lemma [Classic preservation]:** If  $\emptyset \vdash_C e : T$  and  $e \rightarrow_C e'$ , then  $\emptyset \vdash_C e' : T$ .

**Proof:** By induction on the typing derivation.

(T\_VAR) Contradictory— $x$  isn't well typed in an empty context.

(T\_CONST) Contradictory—constants don't step.

(T\_ABS) Contradictory—lambdas don't step.

(T\_OP) By cases on the step taken.

(E\_OP)  $\llbracket op \rrbracket(k_1, \dots, k_n) = k$ ; we assume that  $\mathbf{ty}(op)$  correctly assigns types, i.e., if  $\mathbf{ty}(op) = \{x:B_1 \mid e'_1\} \rightarrow \dots \rightarrow \{x:B_n \mid e'_n\} \rightarrow \{x:B \mid e\}$ , then  $e[k/x] \rightarrow_C^* \mathbf{true}$  and  $\vdash_C \{x:B \mid e\}$ . We can therefore conclude that  $\emptyset \vdash_C k : \{x:B \mid e\}$  by T\_CONST.

(E\_OPINNER) By the IH and T\_OP.

(E\_OPRAISE) We assume that  $\mathbf{ty}(op)$  only assigns well formed types, so  $\emptyset \vdash_C \uparrow l : T$ .

(T\_APP) By cases on the step taken.

(E\_BETA) We know that  $x:T_1 \vdash_C e_1 : T_2$  and  $\emptyset \vdash_C e_2 : T_1$ ; we are done by substitution (Lemma A.2).

(E\_UNWRAP) By inversion of  $\vdash_C \emptyset \parallel T_{11} \rightarrow T_{12} \Rightarrow T_{21} \rightarrow T_{22}$ , we find well formedness judgments  $\vdash_C T_{11}$  and  $\vdash_C T_{12}$  and  $\vdash_C T_{21}$  and  $\vdash_C T_{22}$  and similarity judgments  $\vdash T_{11} \parallel T_{21}$  and  $\vdash T_{12} \parallel T_{22}$ ; by symmetry (Lemma A.6),  $\vdash T_{21} \parallel T_{11}$ . Noting that  $\mathbf{dom}(\bullet) = \mathbf{cod}(\bullet) = \bullet$ , we can apply A\_NONE, finding  $\vdash_C \bullet \parallel T_{21} \Rightarrow T_{11}$  and  $\vdash_C \bullet \parallel T_{12} \Rightarrow T_{22}$ . We can then apply T\_CAST, T\_APP, and assumptions to find  $\emptyset \vdash_C \langle T_{12} \xrightarrow{\bullet} T_{22} \rangle^l (e_1 (\langle T_{21} \xrightarrow{\bullet} T_{11} \rangle^l e_2)) : T_{22}$ .

(E\_APPL) By T\_APP and the IH.

(E\_APPR) By T\_APP and the IH.

(E\_APPRAISEL) By regularity,  $\vdash_C T_2$ , so we are done by T\_BLAKE.

(E\_APPRAISER) By regularity,  $\vdash_C T_2$ , so we are done by T\_BLAKE.

(T\_CAST) By cases on the step taken.

(E\_CHECKNONE) We have  $\vdash_C \Gamma$  and  $\vdash_C \{x:B \mid e_2\}$  and  $\mathbf{ty}(k) = B$  by inversion and  $e_2[k/x] \rightarrow_C^* e_2[k/x]$  by reflexivity. By substitution (and T\_CONST, to find  $\emptyset \vdash_C k : \{x:B \mid \mathbf{true}\}$ ), we find  $\emptyset \vdash_C e_2[k/x] : \{x:\mathbf{Bool} \mid \mathbf{true}\}$ . We can now apply T\_CHECK, and are done.

(E\_CASTINNERC) By T\_CAST and the IH.

(E\_CASTRAISE) We have by assumption that  $\vdash_C T_2$ , so we are done by T\_BLAKE.

(T\_BLAKE) Contradictory—blame doesn't step.

(T\_CHECK) By cases on the step taken.

(E\_CHECKOK) Since  $\vdash_C \emptyset$  and  $\mathbf{ty}(k) = B$  and  $\vdash_C \{x:B \mid e_1\}$  and  $e_1[k/x] \rightarrow_C^* \mathbf{true}$ , we can apply T\_CONST to find  $\emptyset \vdash_C k : \{x:B \mid e_1\}$ .

(E\_CHECKFAIL) Since  $\vdash_C \emptyset$  and  $\vdash_C \{x:B \mid e_1\}$ , T\_BLAKE shows  $\emptyset \vdash_C \uparrow l : \{x:B \mid e_1\}$ .

(E\_CHECKINNER) By T\_CHECK and the IH.

(E\_CHECKRAISE) As for E\_CHECKFAIL—the differing label doesn't matter.

□

### A.3 Forgetful type soundness

Just as we did for classic  $\lambda_H$  in Appendix A.2, we reuse the theorems from Appendix A.1. Note that if  $e$  is a value in forgetful  $\lambda_H$ , it's also a value in classic  $\lambda_H$ , i.e.,  $\mathbf{val}_F e$  implies  $\mathbf{val}_C e$ .

#### A.16 Lemma [Forgetful determinism]:

If  $e \rightarrow_F e_1$  and  $e \rightarrow_F e_2$  then  $e_1 = e_2$ .

**Proof:** By induction on the first evaluation derivation. □

#### A.17 Lemma [Forgetful canonical forms]:

If  $\emptyset \vdash_F e : T$  and  $\mathbf{val}_F e$  then:

- If  $T = \{x:B \mid e'\}$ , then  $e = k$  and  $\mathbf{ty}(k) = B$  and  $e'[k/x] \rightarrow_F^* \mathbf{true}$ .
- If  $T = T_1 \rightarrow T_2$ , then either  $e = \lambda x:T. e'$  or  $e = \langle T_{11} \rightarrow T_{12} \xrightarrow{\bullet} T_{21} \rightarrow T_{22} \rangle^l \lambda x:T_{11}. e'$ .

**Proof:** By inspection of the rules: T\_CONST is the only rule that types values at base types; T\_ABS and T\_CAST are the only rules that type values at function types. □

#### A.18 Lemma [Forgetful progress]:

If  $\emptyset \vdash_F e : T$ , then either:

1.  $\mathbf{result}_F e$  is a result, i.e.,  $e = \uparrow l$  or  $\mathbf{val}_F e$ ; or
2. there exists an  $e'$  such that  $e \rightarrow_F e'$ .

**Proof:** By induction on the typing derivation.

(T\_VAR) A contradiction— $x$  isn't well typed in the empty environment.

(T\_CONST)  $e = k$  is a result.

(T\_ABS)  $e = \lambda x:T. e'$  is a result.

(T\_OP) We know that  $\text{ty}(op)$  is a first order type  $\{x:B_1 \mid e'_1\} \rightarrow \dots \rightarrow \{x:B_n \mid e'_n\} \rightarrow T$ . From left to right, we apply the IH on  $e_i$ . If  $e_i$  is a result, there are two cases: either  $e_i = \uparrow l$ , and we step by E\_OPRAISE; or  $e_i = k_i$ , since constants are the only values at base types by canonical forms (Lemma A.17), and we continue on to the next  $e_i$ . If any of the  $e_i$  step, we know that all of terms before them are values, so we can step by E\_OPINNER. If all of the  $e_i$  are constants  $k_i$ , then  $\emptyset \vdash_F k_i : \{x:B_i \mid e'_i\}$ , and so  $e'_i[k_i/x] \rightarrow_F^* \text{true}$ . Therefore  $\llbracket op \rrbracket(k_1, \dots, k_n)$  is defined, and we can step by E\_OP.

(T\_APP) By the IH on  $e_1$ , we know that  $e_1$  is blame, is a value, or steps to some  $e'_1$ . In the first case, we take a step by E\_APPRAISEL. In the latter, we take a step by E\_APPL.

If  $\text{val}_C e_1$ , then we can apply the IH on  $e_2$ , which is blame, is a value, or steps to some  $e'_2$ .

If  $e_2$  is blame, we step by E\_APPRAISER. Otherwise, we must go by cases. Since  $\emptyset \vdash_F e_1 : T_1 \rightarrow T_2$  and  $\text{val}_F e_1$ , by canonical forms (Lemma A.17) we know that  $e_1$  can only be an abstraction, a wrapped abstraction, or a cast.

( $e_1 = \lambda x:T_1. e'_1$ ) If  $\text{val}_F e_2$ , we step to  $e'_1[e_2/x]$  by E\_BETA. If  $e_2 \rightarrow_F e'_2$ , we step by E\_APPR (since  $e_1$  isn't a cast).

( $e_1 = \langle T_{11} \rightarrow T_{12} \Rightarrow T_{21} \rightarrow T_{22} \rangle^l e'_1$ ) If  $\text{val}_F e_2$ , we step to  $\langle T_{12} \Rightarrow T_{22} \rangle^l (e'_1 (\langle T_{21} \Rightarrow T_{11} \rangle^l e_2))$  by E\_UNWRAP (noting that the annotation is  $\bullet$ ). If  $e_2 \rightarrow_F e'_2$ , we step by E\_APPR (since  $e_1$  isn't a cast).

(T\_CAST) If  $e' = \uparrow l'$ , then we step by E\_CASTRAISE. If  $\text{val}_F e_2$ , then we have  $\vdash_F \bullet \parallel T_1 \Rightarrow T_2$ ; by inversion  $\vdash T_1 \parallel T_2$ . By inversion, one of the following cases adheres:

( $T_i = \{x:B \mid e_{1i}\}$ ) By canonical forms (Lemma A.17),  $e_2 = k$ . We can step to  $\langle \{x:B \mid e_{12}\}, e_{12}[k/x], k \rangle^l$  by E\_CHECKNONE.

( $T_i = T_{i1} \rightarrow T_{i2}$ ) By canonical forms (Lemma A.17),  $e_2$  is a lambda or a function proxy. In the latter case, we step by E\_CASTMERGE; either way, we have a function proxy, which is a value.

If  $e_2$  isn't a value or blame, then  $e_2 \rightarrow_F e'_2$ . If  $e_2 \neq \langle T_{31} \rightarrow T_{32} \Rightarrow T_{11} \rightarrow T_{12} \rangle^l e''_2$ , then we can step by E\_CASTINNER. If  $e_2$  is in fact an application of a cast, we step by E\_CASTMERGE.

(T\_BLADE)  $\uparrow l$  is a result.

(T\_CHECK) By the IH on  $e_2$ , we know that  $e_2$  is  $\uparrow l'$ , is a value, or takes a step to some  $e'_2$ . In the first case, we step to  $\uparrow l'$  by E\_CHECKRAISE. In the last case, we step by E\_CHECKINNER. If  $\text{val}_F e_2$ , by canonical forms (Lemma A.17) we know that  $e_2$  is a  $k$  such that  $\text{ty}(k) = \text{Bool}$ , i.e.,  $e_2$  is either **true** or **false**. In the former case, we step to  $k$  by E\_CHECKOK; in the latter case, we step to  $\uparrow l$  by E\_CHECKFAIL.

□

**A.19 Lemma [Forgetful preservation]:** If  $\emptyset \vdash_F e : T$  and  $e \rightarrow_F e'$  then  $\emptyset \vdash_F e' : T$ .

**Proof:** By induction on the typing derivation.

(T\_VAR) Contradictory—we assumed  $e$  was well typed in an empty context.

(T\_CONST) Contradictory— $k$  is a value and doesn't step.

(T\_ABS) Contradictory— $\lambda x:T_1. e'$  is a value and doesn't step.

(T\_OP) By cases on the step taken.

(E\_OP)  $\llbracket op \rrbracket(k_1, \dots, k_n) = k$ ; we assume that  $\text{ty}(op)$  correctly assigns types, i.e., if  $\text{ty}(op) = \{x:B_1 \mid e'_1\} \rightarrow \dots \rightarrow \{x:B_n \mid e'_n\} \rightarrow \{x:B \mid e\}$ , then  $e[k/x] \rightarrow_F^* \text{true}$  and  $\vdash_F \{x:B \mid e\}$ . We can therefore conclude that  $\emptyset \vdash_F k : \{x:B \mid e\}$  by T\_CONST.

(E\_OPINNER) By the IH and T\_OP.

(E\_OPRAISE) We assume that  $\text{ty}(op)$  only assigns well formed types, so  $\emptyset \vdash_F \uparrow l : T$ .

(T\_APP) By cases on the step taken.

(E\_BETA) We know that  $x:T_1 \vdash_F e_1 : T_2$  and  $\emptyset \vdash_F e_2 : T_1$ ; we are done by substitution (Lemma A.2).

(E\_UNWRAP) By inversion of  $\vdash_F \emptyset \parallel T_{11} \rightarrow T_{12} \Rightarrow T_{21} \rightarrow T_{22}$ , we find well formedness judgments  $\vdash_F T_{11}$  and  $\vdash_F T_{12}$  and  $\vdash_F T_{21}$  and  $\vdash_F T_{22}$  and similarity judgments  $\vdash T_{11} \parallel T_{21}$  and  $\vdash T_{12} \parallel T_{22}$ ; by symmetry (Lemma A.6),  $\vdash T_{21} \parallel T_{11}$ . Noting that  $\text{dom}(\bullet) = \text{cod}(\bullet) = \bullet$ , we can apply A\_NONE to find derivations  $\vdash_F \bullet \parallel T_{21} \Rightarrow T_{11}$  and  $\vdash_F \bullet \parallel T_{12} \Rightarrow T_{22}$ . We can then apply T\_CAST, T\_APP, and assumptions to find  $\emptyset \vdash_F \langle T_{12} \Rightarrow T_{22} \rangle^l (e_1 (\langle T_{21} \Rightarrow T_{11} \rangle^l e_2)) : T_{22}$ .

(E\_APPL) By T\_APP and the IH.

(E\_APPR) By T\_APP and the IH.

(E\_APPRAISEL) By regularity,  $\vdash_F T_2$ , so we are done by T\_BLADE.

(E\_APPRAISER) By regularity,  $\vdash_F T_2$ , so we are done by T\_BLADE.

(T\_CAST) By cases on the step taken.

(E\_CHECKNONE) We have  $\vdash_F \Gamma$  and  $\vdash_F \{x:B \mid e_2\}$  and  $\text{ty}(k) = B$  by inversion and  $e_2[k/x] \longrightarrow_F^* e_2[k/x]$  by reflexivity. By substitution (and T\_CONST, to find  $\emptyset \vdash_F k : \{x:B \mid \text{true}\}$ ), we find  $\emptyset \vdash_F e_2[k/x] : \{x:\text{Bool} \mid \text{true}\}$ . We can now apply T\_CHECK, and are done.

(E\_CASTINNER) By T\_CAST and the IH.

(E\_CASTMERGE) We can combine the two T\_CAST derivations (using Lemma A.10 and the fact that  $a = \bullet$  to find similarity). Then we are done by T\_CAST.

(E\_CASTRAISE) We have  $\vdash_F T_2$  by assumption, so we are done by T\_BLAKE.

(T\_BLAKE) Contradictory— $\uparrow l$  is a result and doesn't step.

(T\_CHECK) By cases on the step taken.

(E\_CHECKOK) Since  $\vdash_F \emptyset$  and  $\text{ty}(k) = B$  and  $\vdash_F \{x:B \mid e_1\}$  and  $e_1[k/x] \longrightarrow_F^* \text{true}$ , we can apply T\_CONST to find  $\emptyset \vdash_F k : \{x:B \mid e_1\}$ .

(E\_CHECKFAIL) Since  $\vdash_F \emptyset$  and  $\vdash_F \{x:B \mid e_1\}$ , T\_BLAKE shows  $\emptyset \vdash_F \uparrow l : \{x:B \mid e_1\}$ .

(E\_CHECKINNER) By T\_CHECK and the IH.

(E\_CHECKRAISE) As for E\_CHECKFAIL—the differing label doesn't matter.

□

In addition to showing type soundness, we prove that a source program (Definition 3.1) is well typed with  $m = F$  iff it is well typed with  $m = C$ .

#### A.20 Lemma [Source program typing for forgetful $\lambda_H$ ]:

Source programs are well typed in  $C$  iff they are well typed in  $F$ , i.e.:

- $\Gamma \vdash_C e : T$  as a source program iff  $\Gamma \vdash_F e : T$  as a source program.
- $\vdash_C T$  as a source program iff  $\vdash_F T$  as a source program.
- $\vdash_C \Gamma$  as a source program iff  $\vdash_F \Gamma$  as a source program.

**Proof:** By mutual induction on  $e$ ,  $T$ , and  $\Gamma$ . Since all of the rules are syntax directed, we use the rule names for cases (but prove both directions at once).

#### Expressions $e$

(T\_VAR) By the IH on  $\Gamma$  and T\_VAR.

(T\_CONST) By the IH on  $\Gamma$  and T\_CONST, noting that  $\text{true} \longrightarrow_m^* \text{true}$  in every mode  $m$ .

(T\_ABS) By the IH on  $T_1$  and  $e_{12}$  and T\_ABS.

(T\_OP) By the IHs on the arguments  $e_i$  and T\_OP.

(T\_APP) By the IHs on  $e_1$  and  $e_2$  and T\_APP.

(T\_CAST) By the IHs on  $T_1$  and  $T_2$  and T\_CAST, noting that similarity holds irrespective of modes and that the annotation is  $\bullet$ .

(T\_BLAKE) Contradictory—doesn't occur in source programs.

(T\_CHECK) Contradictory—doesn't occur in source programs.

#### Types $T$

(WF\_BASE) Immediately true—WF\_BASE is an axiom.

(WF\_REFINE) By the IH on  $e$  and WF\_REFINE.

(WF\_FUN) By the IHs on  $T_1$  and  $T_2$  and WF\_FUN.

#### Contexts $\Gamma$

(WF\_EMPTY) Immediately true—WF\_EMPTY is an axiom.

(WF\_EXTEND) By the IHs on  $\Gamma$  and  $T$  and WF\_EXTEND.

□

## A.4 Heedful type soundness

**A.21 Lemma [Heedful canonical forms]:** If  $\emptyset \vdash_H e : T$  and  $\text{val}_H e$  then:

- If  $T = \{x:B \mid e'\}$ , then  $e = k$  and  $\text{ty}(k) = B$  and  $e'[e/x] \rightarrow_H^* \text{true}$ .
- If  $T = T_1 \rightarrow T_2$ , then either  $e = \lambda x:T. e'$  or  $e = \langle T_{11} \rightarrow T_{12} \xrightarrow{S} T_{21} \rightarrow T_{22} \rangle^l \lambda x:T_{11}. e'$ .

**Proof:** By inspection of the rules: T\_CONST is the only rule that types values at base types; T\_ABS and T\_CAST are the only rules that type values at function types. Note that now our proxies may have type sets in them.  $\square$

**A.22 Lemma [Heedful progress]:** If  $\emptyset \vdash_H e : T$ , then either:

1.  $\text{result}_H e$ , i.e.,  $e = \uparrow l$  or  $\text{val}_H e$ ; or
2. there exists an  $e'$  such that  $e \rightarrow_H e'$ .

**Proof:** By induction on the typing derivation.

(T\_VAR) A contradiction— $x$  isn't well typed in the empty environment.

(T\_CONST)  $e = k$  is a result by V\_CONST and R\_VAL.

(T\_ABS)  $e = \lambda x:T. e'$  is a result by V\_ABS and R\_VAL.

(T\_OP) We know that  $\text{ty}(op)$  is a first order type  $\{x:B_1 \mid e'_1\} \rightarrow \dots \rightarrow \{x:B_n \mid e'_n\} \rightarrow T$ . From left to right, we apply the IH on  $e_i$ . If  $e_i$  is a result, there are two cases: either  $e_i = \uparrow l$ , and we step by E\_OPRAISE; or  $e_i = k_i$ , since constants are the only values at base types by canonical forms (Lemma A.21), and we continue on to the next  $e_i$ . If any of the  $e_i$  step, we know that all of terms before them are values, so we can step by E\_OPINNER. If all of the  $e_i$  are constants  $k_i$ , then  $\emptyset \vdash_E k_i : \{x:B_i \mid e'_i\}$ , and so  $e'_i[k_i/x] \rightarrow_H^* \text{true}$ . Therefore  $\llbracket op \rrbracket(k_1, \dots, k_n)$  is defined, and we can step by E\_OP.

(T\_APP) By the IH on  $e_1$ , we know that  $e_1$  is blame, is a value, or steps to some  $e'_1$ . In the first case, we take a step by E\_APPRAISEL. In the latter, we take a step by E\_APPL.

If  $\text{val}_H e_1$ , then we can apply the IH on  $e_2$ , which is blame, is a value, or steps to some  $e'_2$ . The first and last cases are as before, using E\_APPRAISER and E\_APPR.

If  $\text{val}_H e_2$ , we must go by cases on the shape of  $e_1$ . Since  $\emptyset \vdash_H e_1 : T_1 \rightarrow T_2$ , by canonical forms (Lemma A.21) we know that  $e_1$  can only be an abstraction, a wrapped abstraction, or a cast.

( $e_1 = \lambda x:T_1. e'_1$ ) We step to  $e'_1[e_2/x]$  by E\_BETA.

( $e_1 = \langle T_{11} \rightarrow T_{12} \xrightarrow{S} T_{21} \rightarrow T_{22} \rangle^l e'_1$ ) We step to  $\langle T_{12} \xrightarrow{\text{cod}(S)} T_{22} \rangle^l (e'_1 (\langle T_{21} \xrightarrow{\text{dom}(S)} T_{11} \rangle^l e_2))$  by E\_UNWRAP.

(T\_CAST) If the annotation is  $\bullet$ , we step by E\_TYPESET. We have  $e = \langle T_1 \xrightarrow{S} T_2 \rangle^l e_1$ . If  $e_1$  is blame, then we step by E\_CASTRAISE. If  $e_1$  is a value, then we go by cases on  $\text{val}_H e_1$ :

(V\_CONST) The case must be between refinements, and we step by E\_CHECKSET or E\_CHECKEMPTY.

(V\_ABS) The cast must be between function types, and we have a value.

(V\_PROXYH) The cast must be between function types, and we step by E\_CASTMERGE.

Finally, it may be the case that  $e_1 \rightarrow_H e'_1$ . If  $e_1 \neq \langle T_3 \xrightarrow{S} T_1 \rangle^l e''_1$ , then we step by E\_CASTINNER. On the other hand, if  $e_1$  is a cast term, we step by E\_CASTMERGE.

(T\_BLAZE)  $\uparrow l$  is a result by R\_BLAZE.

(T\_CHECK) By the IH on  $e_2$ , we know that  $e_2$  is  $\uparrow l'$ , is a value by  $\text{val}_H e_2$ , or takes a step to some  $e'_2$ . In the first case, we step to  $\uparrow l'$  by E\_CHECKRAISE. In the last case, we step by E\_CHECKINNER. If  $\text{val}_E e_2$ , by canonical forms (Lemma A.21) we know that  $e_2$  is a  $k$  such that  $\text{ty}(k) = \text{Bool}$ , i.e.,  $e_2$  is either **true** or **false**. In the former case, we step to  $k$  by E\_CHECKOK; in the latter case, we step to  $\uparrow l$  by E\_CHECKFAIL.

$\square$

Before proving preservation, we must establish some properties about type sets: type sets as merged by E\_CASTMERGE are well formed; the  $\text{dom}$  and  $\text{cod}$  operators take type sets of function types and produce well formed type sets.

**A.23 Lemma [Merged type sets are well formed]:** If  $\vdash_H S_1 \parallel T_1 \Rightarrow T_2$  and  $\vdash_H S_2 \parallel T_2 \Rightarrow T_3$  then  $\vdash_H (S_1 \cup S_2 \cup \{T_2\}) \parallel T_1 \Rightarrow T_3$ .

**Proof:** By transitivity of similarity, we have  $\vdash T_1 \parallel T_3$ . We have  $\vdash_H T_1$  and  $\vdash_H T_3$  from each of the A\_TYPESET derivations, so it remains to show the premises for each  $T \in S$ .

Let  $T \in (S_1 \cup S_2 \cup \{T_2\})$ . We have  $\vdash T \parallel T_1$  and  $\vdash_H T$  (a) by assumption and symmetry (Lemma A.6) if  $T = T_2$ ; and (b) by A\_TYPESET and symmetry and transitivity (Lemma A.7) if  $T \in S_1 \cup S_2$ . We can therefore apply A\_TYPESET, and we are done.  $\square$



**A.24 Lemma [Domain type set well formedness]:** If  $\vdash_{\mathcal{H}} \mathcal{S} \parallel T_{11} \rightarrow T_{12} \Rightarrow T_{21} \rightarrow T_{22}$  then  $\vdash_{\mathcal{H}} \text{dom}(\mathcal{S}) \parallel T_{21} \Rightarrow T_{11}$ .

**Proof:** First, observe that for every  $T \in \mathcal{S}$ , we know that  $\vdash T \parallel T_{11} \rightarrow T_{12}$ , so each  $T_i = T_{i1} \rightarrow T_{i2}$  by inversion. This means that  $\text{dom}(\mathcal{S})$  is well defined.

By inversion of similarity and type well formedness, we have  $\vdash T_{11} \parallel T_{21}$  and  $\vdash_{\mathcal{H}} T_{11}$  and  $\vdash_{\mathcal{H}} T_{21}$ . By symmetry of similarity, we have  $\vdash T_{21} \parallel T_{11}$  (Lemma A.6).

Let  $T_{i1} \in \text{dom}(\mathcal{S})$  by given. We know that there exists some  $T_{i2}$  such that  $T_{i1} \rightarrow T_{i2} \in \mathcal{S}$  and  $\vdash T_{i1} \rightarrow T_{i2} \parallel T_{11} \rightarrow T_{12}$  and  $\vdash_{\mathcal{H}} T_{i1} \rightarrow T_{i2}$ . By inversion, we find  $\vdash T_{i1} \parallel T_{11}$  and  $\vdash_{\mathcal{H}} T_{i1}$ . By transitivity of similarity (Lemma A.7), we have  $\vdash T_{i1} \parallel T_{21}$ , and we are done by A\_TYPESET.  $\square$

**A.25 Lemma [Codomain type set well formedness]:** If  $\vdash_{\mathcal{H}} \mathcal{S} \parallel T_{11} \rightarrow T_{12} \Rightarrow T_{21} \rightarrow T_{22}$  then  $\vdash_{\mathcal{H}} \text{cod}(\mathcal{S}) \parallel T_{12} \Rightarrow T_{22}$ .

**Proof:** First, observe that for every  $T \in \mathcal{S}$ , we know that  $\vdash T \parallel T_{11} \rightarrow T_{12}$ , so each  $T_i = T_{i1} \rightarrow T_{i2}$  by inversion. This means that  $\text{dom}(\mathcal{S})$  is well defined.

By inversion of similarity and type well formedness, we have  $\vdash T_{12} \parallel T_{22}$  and  $\vdash_{\mathcal{H}} T_{12}$  and  $\vdash_{\mathcal{H}} T_{22}$ .

Let  $T_{i2} \in \text{cod}(\mathcal{S})$  by given. We know that there exists some  $T_{i1}$  such that  $T_{i1} \rightarrow T_{i2} \in \mathcal{S}$  and  $\vdash T_{i1} \rightarrow T_{i2} \parallel T_{12} \rightarrow T_{12}$  and  $\vdash_{\mathcal{H}} T_{i1} \rightarrow T_{i2}$ . By inversion, we find  $\vdash T_{i2} \parallel T_{12}$  and  $\vdash_{\mathcal{H}} T_{i2}$ . We are done by A\_TYPESET.  $\square$

**A.26 Lemma [Heedful preservation]:** If  $\emptyset \vdash_{\mathcal{H}} e : T$  and  $e \rightarrow_{\mathcal{H}} e'$  then  $\emptyset \vdash_{\mathcal{H}} e' : T$ .

**Proof:** By induction on the typing derivation.

(T\_VAR) Contradictory—we assumed  $e$  was well typed in an empty context.

(T\_CONST) Contradictory— $k$  is a value and doesn't step.

(T\_ABS) Contradictory— $\lambda x:T_1. e'$  is a value and doesn't step.

(T\_OP) By cases on the step taken.

(E\_OP)  $\llbracket op \rrbracket (k_1, \dots, k_n) = k$ ; we assume that  $\text{ty}(op)$  correctly assigns types, i.e., if  $\text{ty}(op) = \{x:B_1 \mid e'_1\} \rightarrow \dots \rightarrow \{x:B_n \mid e'_n\} \rightarrow \{x:B \mid e\}$ , then  $e[k/x] \rightarrow_{\mathcal{H}}^* \text{true}$  and  $\vdash_{\mathcal{H}} \{x:B \mid e\}$ . We can therefore conclude that  $\emptyset \vdash_{\mathcal{H}} k : \{x:B \mid e\}$  by T\_CONST.

(E\_OPINNER) By the IH and T\_OP.

(E\_OPRAISE) We assume that  $\text{ty}(op)$  only assigns well formed types, so  $\emptyset \vdash_{\mathcal{H}} \uparrow l : T$ .

(T\_APP) By cases on the step taken.

(E\_BETA) We know that  $x:T_1 \vdash_{\mathcal{H}} e_1 : T_2$  and  $\emptyset \vdash_{\mathcal{H}} e_2 : T_1$ ; we are done by substitution (Lemma A.2).

(E\_UNWRAP) We have  $e = (\langle T_{11} \rightarrow T_{12} \xRightarrow{\mathcal{S}} T_{21} \rightarrow T_{22} \rangle^l e_1) e_2 \rightarrow_{\mathcal{H}} \langle T_{12} \xRightarrow{\text{cod}(\mathcal{S})} T_{22} \rangle^l (e_1 (\langle T_{21} \xRightarrow{\text{dom}(\mathcal{S})} T_{11} \rangle^l e_2)) = e'$ .

By inversion,  $\vdash T_{11} \rightarrow T_{12} \parallel T_{21} \rightarrow T_{22}$  and  $\vdash_{\mathcal{H}} \mathcal{S} \parallel T_{11} \rightarrow T_{12} \Rightarrow T_{21} \rightarrow T_{22}$ . By Lemma A.24,  $\vdash_{\mathcal{H}} \text{dom}(\mathcal{S}) \parallel T_{21} \Rightarrow T_{11}$ ; by Lemma A.25,  $\vdash_{\mathcal{H}} \text{cod}(\mathcal{S}) \parallel T_{12} \Rightarrow T_{22}$ .

By inversion,  $\emptyset \vdash_{\mathcal{H}} e_2 : T_{21}$ ; so  $\emptyset \vdash_{\mathcal{H}} \langle T_{21} \xRightarrow{\text{dom}(\mathcal{S})} T_{11} \rangle^l e_2 : T_{11}$  by T\_CAST. By inversion,  $\emptyset \vdash_{\mathcal{H}} e_1 : T_{11} \rightarrow T_{12}$ , so  $\emptyset \vdash_{\mathcal{H}} e_1 (\langle T_{21} \xRightarrow{\text{dom}(\mathcal{S})} T_{11} \rangle^l e_2) : T_{12}$  by T\_APP. By the type set typing for  $\text{cod}(\mathcal{S})$ , we can apply T\_CAST to finish the case, typing the whole term at  $T_{22}$ .

(E\_APPL) By T\_APP and the IH.

(E\_APPR) By T\_APP and the IH.

(E\_APPRAISEL) By regularity,  $\vdash_{\mathcal{H}} T_2$ , so we are done by T\_BLAKE.

(E\_APPRAISER) By regularity,  $\vdash_{\mathcal{H}} T_2$ , so we are done by T\_BLAKE.

(T\_CAST) By cases on the step taken.

(E\_TYPESET) By the assumptions, using A\_NONE to derive A\_TYPESET—which holds vacuously, since the set is empty.

(E\_CHECKEMPTY) We have  $\vdash_{\mathcal{H}} \Gamma$  and  $\vdash_{\mathcal{H}} \{x:B \mid e_2\}$  and  $\text{ty}(k) = B$  by inversion and  $e_2[k/x] \rightarrow_{\mathcal{H}}^* e_2[k/x]$  by reflexivity. By substitution (and T\_CONST, to find  $\emptyset \vdash_{\mathcal{H}} k : \{x:B \mid \text{true}\}$ ), we find  $\emptyset \vdash_{\mathcal{H}} e_2[k/x] : \{x:\text{Bool} \mid \text{true}\}$ . We can now apply T\_CHECK, and are done.

(E\_CHECKSET) We have

$$\langle \{x:B \mid e_1\} \xRightarrow{\mathcal{S}} \{x:B \mid e_3\} \rangle^l k \rightarrow_{\mathcal{H}} \langle \{x:B \mid e_2\} \xRightarrow{\mathcal{S} \setminus \{x:B \mid e_2\}} \{x:B \mid e_3\} \rangle^l \langle \{x:B \mid e_2\}, e_2[k/x], k \rangle^l$$

where  $\text{choose}(\mathcal{S}) = \{x:B \mid e_2\}$ , so  $\{x:B \mid e_2\} \in \mathcal{S}$ . We have  $\vdash_{\mathcal{H}} \Gamma$  and  $\vdash_{\mathcal{H}} \{x:B \mid e_2\}$  and  $\text{ty}(k) = B$  by inversion and  $e_2[k/x] \rightarrow_{\mathcal{H}}^* e_2[k/x]$  by reflexivity. By substitution (and T\_CONST, to find  $\emptyset \vdash_{\mathcal{H}} k : \{x:B \mid \text{true}\}$ ), we find  $\emptyset \vdash_{\mathcal{H}} e_2[k/x] : \{x:\text{Bool} \mid \text{true}\}$ . We can now apply T\_CHECK to type the active check at  $\{x:B \mid e_2\}$ .

By inversion of the original type set well formedness derivation,  $\vdash \{x:B \mid e_2\} \parallel \{x:B \mid e_1\}$  and  $\vdash_H \{\{x:B \mid e_2\}\} \cup \mathcal{S} \parallel \{x:B \mid e_1\} \Rightarrow \{x:B \mid e_3\}$ ; by Lemma A.11,  $\vdash_H \mathcal{S} \setminus \{x:B \mid e_2\} \parallel \{x:B \mid e_2\} \Rightarrow \{x:B \mid e_3\}$ . We have  $\vdash \{x:B \mid e_2\} \parallel \{x:B \mid e_3\}$  transitivity of similarity ( $\vdash \{x:B \mid e_1\} \parallel \{x:B \mid e_3\}$  and Lemma A.7). So now we can apply T\_CAST to our T\_CHECK derivation, and we are done.

(E\_CASTINNER) By T\_CAST and the IH.

(E\_CASTMERGE) We have

$$\langle T_2 \xRightarrow{\mathcal{S}_2} T_3 \rangle^{l_2} (\langle T_1 \xRightarrow{\mathcal{S}_1} T_2 \rangle^{l_1} e) \longrightarrow_H \langle T_1 \xRightarrow{(\mathcal{S}_1 \cup \mathcal{S}_2 \cup \{T_2\})} T_3 \rangle^{l_2} e.$$

By Lemma A.23,  $\vdash_H (\mathcal{S}_1 \cup \mathcal{S}_2 \cup \{T_2\}) \parallel T_1 \Rightarrow T_3$ . We already know that  $\emptyset \vdash_H e_1 : T_1$  (by assumption) and  $\vdash_H T_3$  (by inversion of the outer cast's typing derivation), so we can apply T\_CAST to type the resulting merged cast.

(E\_CASTRAISE) We have  $\vdash_H T_2$  by assumption, so we are done by T\_BLAKE.

(T\_BLAKE) Contradictory— $\uparrow l$  is a result and doesn't step.

(T\_CHECK) By cases on the step taken.

(E\_CHECKOK) Since  $\vdash_H \emptyset$  and  $\text{ty}(k) = B$  and  $\vdash_H \{x:B \mid e_1\}$  and  $e_1[k/x] \longrightarrow_H^* \text{true}$ , we can apply T\_CONST to find  $\emptyset \vdash_H k : \{x:B \mid e_1\}$ .

(E\_CHECKFAIL) Since  $\vdash_H \emptyset$  and  $\vdash_H \{x:B \mid e_1\}$ , T\_BLAKE shows  $\emptyset \vdash_H \uparrow l : \{x:B \mid e_1\}$ .

(E\_CHECKINNER) By T\_CHECK and the IH.

(E\_CHECKRAISE) As for E\_CHECKFAIL—the differing label doesn't matter.

□

Just as we did for forgetful  $\lambda_H$  in (Appendix A.3), we show that source programs are well typed heedfully iff they are well typed classically—iff they are well typed forgetfull (Lemma A.20). that is, source programs are valid staring points in any mode.

#### A.27 Lemma [Source program typing for heedful $\lambda_H$ ]:

Source programs are well typed in  $\mathbf{C}$  iff they are well typed in  $\mathbf{H}$ , i.e.:

- $\Gamma \vdash_{\mathbf{C}} e : T$  as a source program iff  $\Gamma \vdash_H e : T$  as a source program.
- $\vdash_{\mathbf{C}} T$  as a source program iff  $\vdash_H T$  as a source program.
- $\vdash_{\mathbf{C}} \Gamma$  as a source program iff  $\vdash_H \Gamma$  as a source program.

**Proof:** By mutual induction on  $e$ ,  $T$ , and  $\Gamma$ . Since all of the rules are syntax directed, we use the rule names for cases (but prove both directions at once).

#### Expressions $e$

(T\_VAR) By the IH on  $\Gamma$  and T\_VAR.

(T\_CONST) By the IH on  $\Gamma$  and T\_CONST, noting that  $\text{true} \longrightarrow_m^* \text{true}$  in every mode  $m$ .

(T\_ABS) By the IH on  $T_1$  and  $e_{12}$  and T\_ABS.

(T\_OP) By the IHs on the arguments  $e_i$  and T\_OP.

(T\_APP) By the IHs on  $e_1$  and  $e_2$  and T\_APP.

(T\_CAST) By the IHs on  $T_1$  and  $T_2$  and T\_CAST, using A\_NONE for the source program.

(T\_BLAKE) Contradictory—doesn't occur in source programs.

(T\_CHECK) Contradictory—doesn't occur in source programs.

#### Types $T$

(WF\_BASE) Immediately true—WF\_BASE is an axiom.

(WF\_REFINE) By the IH on  $e$  and WF\_REFINE.

(WF\_FUN) By the IHs on  $T_1$  and  $T_2$  and WF\_FUN.

#### Contexts $\Gamma$

(WF\_EMPTY) Immediately true—WF\_EMPTY is an axiom.

(WF\_EXTEND) By the IHs on  $\Gamma$  and  $T$  and WF\_EXTEND.

□

## A.5 Eidetic type soundness

**A.28 Lemma [Determinism of eidetic  $\lambda_H$ ]:** If  $e \rightarrow_E e_1$  and  $e \rightarrow_E e_2$  then  $e_1 = e_2$ .

**Proof:** By induction on the first evaluation derivation. In every case, only a single step can be taken.  $\square$

**A.29 Lemma [Eidetic canonical forms]:** If  $\emptyset \vdash_E e : T$  and  $\text{val}_E e$  then:

- If  $T = \{x:B \mid e'\}$ , then  $e = k$  and  $\text{ty}(k) = B$  and  $e'[e/x] \rightarrow_E^* \text{true}$ .
- If  $T = T_{21} \rightarrow T_{22}$ , then either  $e = \lambda x:T. e'$  or  $e = \langle T_{11} \rightarrow T_{12} \xrightarrow{c_1 \mapsto c_2} T_{21} \rightarrow T_{22} \rangle^\bullet \lambda x:T_{11}. e'$ .

**Proof:** By inspection of the rules: T\_CONST is the only rule that types values at base types; T\_ABS and T\_CAST are the only rules that type values at function types. Note that now our proxies use coercions and not types and type sets.  $\square$

**A.30 Lemma [Eidetic progress]:** If  $\emptyset \vdash_E e : T$ , then either:

1.  $\text{result}_E e$ , i.e.,  $e = \uparrow l$  or  $\text{val}_E e$ ; or
2. there exists an  $e'$  such that  $e \rightarrow_E e'$ .

**Proof:** By induction on the typing derivation.

(T\_VAR) A contradiction— $x$  isn't well typed in the empty environment.

(T\_CONST)  $e = k$  is a result by V\_CONST and R\_VAL.

(T\_ABS)  $e = \lambda x:T. e'$  is a result by V\_ABS and R\_VAL.

(T\_OP) We know that  $\text{ty}(op)$  is a first order type  $\{x:B_1 \mid e'_1\} \rightarrow \dots \rightarrow \{x:B_n \mid e'_n\} \rightarrow T$ . From left to right, we apply the IH on  $e_i$ . If  $e_i$  is a result, there are two cases: either  $e_i = \uparrow l$ , and we step by E\_OPRAISE; or  $e_i = k_i$ , since constants are the only values at base types by canonical forms (Lemma A.29), and we continue on to the next  $e_i$ . If any of the  $e_i$  step, we know that all of terms before them are values, so we can step by E\_OPINNER. If all of the  $e_i$  are constants  $k_i$ , then  $\emptyset \vdash_E k_i : \{x:B_i \mid e'_i\}$ , and so  $e'_i[k_i/x] \rightarrow_H^* \text{true}$ . Therefore  $\llbracket op \rrbracket(k_1, \dots, k_n)$  is defined, and we can step by E\_OP.

(T\_APP) By the IH on  $e_1$ , we know that  $e_1$  is blame, is a value, or steps to some  $e'_1$ . In the first case, we take a step by E\_APPRAISEL. In the latter, we take a step by E\_APPPL.

If  $\text{val}_E e_1$ , then we can apply the IH on  $e_2$ , which is blame, is a value, or steps to some  $e'_2$ . The first and last cases are as before, using E\_APPRAISER and E\_APPR.

If  $\text{val}_E e_2$ , we must go by cases on the shape of  $e_1$ . Since  $\emptyset \vdash_E e_1 : T_1 \rightarrow T_2$ , by canonical forms (Lemma A.29) we know that  $e_1$  can only be an abstraction, a wrapped abstraction, or a cast.

( $e_1 = \lambda x:T_1. e'_1$ ) We step to  $e'_1[e_2/x]$  by E\_BETA.

( $e_1 = \langle T_{11} \rightarrow T_{12} \xrightarrow{c_1 \mapsto c_2} T_{21} \rightarrow T_{22} \rangle^\bullet \lambda x:T_{11}. e'_1$ ) We step to  $\langle T_{12} \xrightarrow{c_2} T_{22} \rangle^\bullet ((\lambda x:T_{11}. e'_1) (\langle T_{21} \xrightarrow{c_1} T_{11} \rangle^\bullet e_2))$  by E\_UNWRAP.

(T\_CAST) We step by E\_COERCE to  $\langle T_1 \xrightarrow{\text{coerce}(T_1, T_2, l)} T_2 \rangle^\bullet e$ .

(T\_COERCE) We have  $e = \langle T_1 \xrightarrow{c} T_2 \rangle^\bullet e_1$ . If  $e_1$  is blame, then we step by E\_COERCERAISE. If  $e_1$  is a value, then we go by cases on  $\text{val}_E e_1$ :

(V\_CONST) The case must be between refinements, and we step by E\_COERCERSTACK.

(V\_ABS) The cast must be between function types, and we have a value by V\_PROXYE.

(V\_PROXYE) We step by E\_CASTMERGE.

Finally, it may be the case that  $e_1 \rightarrow_E e'_1$ . If  $e_1 \neq \langle T_3 \xrightarrow{c'} T_1 \rangle^\bullet e''_1$ , then we step by E\_CASTINNEREFFICIENT. On the other hand, if  $e_1$  is a coercion term, we step by E\_CASTMERGE.

(T\_BLAZE)  $\uparrow l$  is a result by R\_BLAZE.

(T\_STACK) By cases on the shape of  $r$ : if  $r = \text{nil}$ , we know that  $s = \checkmark$ . We step by E\_STACKDONE if  $e = k$ ; otherwise, we step by E\_STACKINNER or E\_STACKRAISE. If not, we go by cases on the shape of  $e$ . If  $e = k$ , then we step by E\_STACKPOP. Otherwise, we step by E\_STACKINNER or E\_STACKRAISE.

(T\_CHECK) By the IH on  $e_2$ , we know that  $e_2$  is  $\uparrow l'$ , is a value by  $\text{val}_E e_2$ , or takes a step to some  $e'_2$ . In the first case, we step to  $\uparrow l'$  by E\_CHECKRAISE. In the last case, we step by E\_CHECKINNER. If  $\text{val}_E e_2$ , by canonical forms (Lemma A.29) we know that  $e_2$  is a  $k$  such that  $\text{ty}(k) = \text{Bool}$ , i.e.,  $e_2$  is either **true** or **false**. In the former case, we step to  $k$  by E\_CHECKOK; in the latter case, we step to  $\uparrow l$  by E\_CHECKFAIL.  $\square$

**A.31 Lemma [Extended refinement lists are well formed]:**

If  $\vdash_E \{x:B \mid e\}$  and  $\vdash_E r \parallel \{x:B \mid e_1\} \Rightarrow \{x:B \mid e_2\}$  then  $\vdash_E \{x:B \mid e\}^l \triangleright r \parallel \{x:B \mid e_1\} \Rightarrow \{x:B \mid e_2\}$ .

**Proof:** By cases on the rule used.

(A\_REFINE) All of the premises are immediately restored except in one tricky case. When  $\{x:B \mid e\} \supset \{x:B \mid e'\}$  where  $\{x:B \mid e'\} \in r$  is the only type implying  $\{x:B \mid e_2\}$ . Then  $r \setminus \{x:B \mid e\}$  isn't well formed on its own, but adding  $\{x:B \mid e\}^l$  makes it so by transitivity. If not, then we know that  $r \setminus \{x:B \mid e\}$  is well formed, and so is its extensions by assumption.

We know that there are no duplicates by reflexivity of  $\supset$ .

(A\_FUN) Contradictory.

□

**A.32 Lemma [Merged coercions are well formed]:** If  $\vdash_E c_1 \parallel T_1 \Rightarrow T_2$  and  $\vdash_E c_2 \parallel T_2 \Rightarrow T_3$  then  $\vdash_E c_1 \triangleright c_2 \parallel T_1 \Rightarrow T_3$ .

**Proof:** By induction on  $c_1$ 's typing derivation.

(A\_REFINE) By the IH, Lemma A.31, and A\_REFINE.

(A\_FUN) By the IHs and A\_FUN.

□

**A.33 Lemma [coerce generates well formed coercions]:**

If  $\vdash T_1 \parallel T_2$  then  $\vdash_E \text{coerce}(T_1, T_2, l) \parallel T_1 \Rightarrow T_2$ .

**Proof:** By induction on the similarity derivation.

(S\_REFINE) By A\_REFINE, with  $\text{coerce}(\{x:B \mid e_1\}, \{x:B \mid e_2\}, l) = \{x:B \mid e_2\}^l$ .

(S\_FUN) By A\_FUN and the IHs.

□

**A.34 Lemma [Eidetic preservation]:** If  $\emptyset \vdash_E e : T$  and  $e \longrightarrow_E e'$  then  $\emptyset \vdash_E e' : T$ .

**Proof:** By induction on the typing derivation.

(T\_VAR) Contradictory—we assumed  $e$  was well typed in an empty context.

(T\_CONST) Contradictory— $k$  is a value and doesn't step.

(T\_ABS) Contradictory— $\lambda x.T_1$ .  $e'$  is a value and doesn't step.

(T\_OP) By cases on the step taken.

(E\_OP)  $\llbracket op \rrbracket (k_1, \dots, k_n) = k$ ; we assume that  $\text{ty}(op)$  correctly assigns types, i.e., if  $\text{ty}(op) = \{x:B_1 \mid e'_1\} \rightarrow \dots \rightarrow \{x:B_n \mid e'_n\} \rightarrow \{x:B \mid e\}$ , then  $e[k/x] \longrightarrow_E^* \text{true}$  and  $\vdash_E \{x:B \mid e\}$ . We can therefore conclude that  $\emptyset \vdash_E k : \{x:B \mid e\}$  by T\_CONST.

(E\_OPINNER) By the IH and T\_OP.

(E\_OPRAISE) We assume that  $\text{ty}(op)$  only assigns well formed types, so  $\emptyset \vdash_E \uparrow l : T$ .

(T\_APP) By cases on the step taken.

(E\_BETA) We know that  $x:T_1 \vdash_E e_1 : T_2$  and  $\emptyset \vdash_E e_2 : T_1$ ; we are done by substitution (Lemma A.2).

(E\_UNWRAP) We have  $e = (\langle T_{11} \rightarrow T_{12} \xrightarrow{c_1} T_{21} \rightarrow T_{22} \rangle^\bullet e_1) e_2 \longrightarrow_E \langle T_{12} \xrightarrow{c_2} T_{22} \rangle^\bullet (e_1 (\langle T_{21} \xrightarrow{c_1} T_{11} \rangle^\bullet e_2)) = e'$ .

By inversion,  $\vdash_E c_1 \parallel T_{21} \Rightarrow T_{11}$  and  $\vdash_E c_2 \parallel T_{12} \Rightarrow T_{22}$ . We are done by T\_COERCE and T\_APP.

(E\_APPL) By T\_APP and the IH.

(E\_APPR) By T\_APP and the IH.

(E\_APPRAISEL) By regularity,  $\vdash_E T_2$ , so we are done by T\_BLAKE.

(E\_APPRAISER) By regularity,  $\vdash_E T_2$ , so we are done by T\_BLAKE.

(T\_CAST) If the annotation is  $\bullet$ , we step by E\_COERCE, which is well typed by T\_COERCE (using Lemma A.33). Otherwise, by cases on the step taken.

(E\_COERCESTACK) We have  $\vdash_E \Gamma$  and  $\vdash_E \{x:B \mid e_2\}$  and  $\text{ty}(k) = B$  by inversion. The quantification over  $r$  is also by inversion, of coercion well formedness. Since  $s = ?$ , we can find  $\{x:B \mid e\} \in r$  such that  $\{x:B \mid e\} \supset \{x:B \mid e_2\}$  by that same well formedness derivation. So: by T\_STACK.

(E\_COERCEINNER) By T\_COERCE and the IH.

(E\_CASTMERGE) We have

$$\begin{aligned} & \langle T_2 \xRightarrow{c_2} T_3 \rangle^\bullet \cdot (\langle T_1 \xRightarrow{c_1} T_2 \rangle^\bullet e) \longrightarrow_E \\ & \langle T_1 \xRightarrow{c_1 \triangleright c_2} T_3 \rangle^\bullet e. \end{aligned}$$

By Lemma A.32,  $\vdash_E c_1 \triangleright c_2 \parallel T_1 \Rightarrow T_3$ . We already know that  $\emptyset \vdash_E e_1 : T_1$  (by assumption) and  $\vdash_E T_3$  (by inversion of the outer cast's typing derivation), so we can apply T\_COERCE to type the resulting merged coercion.

(E\_COERCERAISE) We have  $\vdash_E T_2$  by assumption, so we are done by T\_BLAKE.

(T\_STACK) By cases on the step taken.

(E\_STACKDONE) We know by assumption that  $\vdash_E \{x:B \mid e\}$  and  $e[k/x] \longrightarrow_E^* \text{true}$ , so by T\_CONST.

(E\_STACKPOP) We have  $\vdash_E \Gamma$  and  $\vdash_E \{x:B \mid e_2\}$  and  $\text{ty}(k) = B$  by inversion. The quantification over  $r$  is also by inversion, of coercion well formedness.

If  $\{x:B \mid e'\} \supset \{x:B \mid e\}$ , then our new status is  $\checkmark$  and we enter a checking form—so the reduction  $\langle \{x:B \mid e'\}, e'[k/x], k \rangle^l \longrightarrow_E^* e_2$  holds by reflexivity.

If not, then our status is whatever it was before. If it was  $\checkmark$ , then that is because we either (a) already knew that  $e_1[k/x] \longrightarrow_E^* \text{true}$  or because  $\langle \{x:B \mid e'\}, e'[k/x], k \rangle^l \longrightarrow_E^* k$  for some  $\{x:B \mid e'\} \supset \{x:B \mid e_1\}$ —which implies that  $e_1[k/x] \longrightarrow_E^* \text{true}$  by adequacy of  $\supset$ . So if  $s = \checkmark$ , our side condition is covered. If  $s = ?$ , then we know that  $\{x:B \mid e_1\}$  remains to be checked, and some  $\{x:B \mid e'\} \supset \{x:B \mid e_1\}$  is in  $r$ .

We can type the active check using our assumptions, where  $e_2[k/x] \longrightarrow_E^* e_2[k/x]$  by reflexivity. By substitution (and T\_CONST, to find  $\emptyset \vdash_E k : \{x:B \mid \text{true}\}$ ), we find  $\emptyset \vdash_E e_2[k/x] : \{x:\text{Bool} \mid \text{true}\}$ . We can now apply T\_CHECK, and can then apply an outer T\_STACK.

E\_STACKINNER By T\_STACK and the IH. If  $s = \checkmark$ , we need to extend the evaluation derivation by one step.

E\_STACKRAISE We have  $\vdash_E \{x:B \mid e\}$  and  $\vdash_E \Gamma$  already, so by T\_BLAKE.

(T\_BLAKE) Contradictory— $\uparrow l$  is a result and doesn't step.

(T\_CHECK) By cases on the step taken.

(E\_CHECKOK) Since  $\vdash_E \emptyset$  and  $\text{ty}(k) = B$  and  $\vdash_E \{x:B \mid e_1\}$  and  $e_1[k/x] \longrightarrow_E^* \text{true}$ , we can apply T\_CONST to find  $\emptyset \vdash_E k : \{x:B \mid e_1\}$ .

(E\_CHECKFAIL) Since  $\vdash_E \emptyset$  and  $\vdash_E \{x:B \mid e_1\}$ , T\_BLAKE shows  $\emptyset \vdash_E \uparrow l : \{x:B \mid e_1\}$ .

(E\_CHECKINNER) By T\_CHECK and the IH.

(E\_CHECKRAISE) As for E\_CHECKFAIL—the differing label doesn't matter.

□

**A.35 Lemma [Source program typing for eidetic  $\lambda_H$ ]:** Source programs are well typed in C iff they are well typed in E, i.e.:

- $\Gamma \vdash_C e : T$  as a source program iff  $\Gamma \vdash_E e : T$  as a source program.
- $\vdash_C T$  as a source program iff  $\vdash_E T$  as a source program.
- $\vdash_C \Gamma$  as a source program iff  $\vdash_E \Gamma$  as a source program.

**Proof:** By mutual induction on  $e$ ,  $T$ , and  $\Gamma$ . Since all of the rules are syntax directed, we use the rule names for cases (but prove both directions at once).

**Expressions  $e$**

(T\_VAR) By the IH on  $\Gamma$  and T\_VAR.

(T\_CONST) By the IH on  $\Gamma$  and T\_CONST, noting that  $\text{true} \longrightarrow_m^* \text{true}$  in every mode  $m$ .

(T\_ABS) By the IH on  $T_1$  and  $e_{12}$  and T\_ABS.

(T\_OP) By the IHs on the arguments  $e_i$  and T\_OP.

(T\_APP) By the IHs on  $e_1$  and  $e_2$  and T\_APP.

(T\_CAST) By the IHs on  $T_1$  and  $T_2$  and T\_CAST, noting that similarity holds irrespective of modes and that the annotation is  $\bullet$ .

(T\_BLAKE) Contradictory—doesn't occur in source programs.

(T\_CHECK) Contradictory—doesn't occur in source programs.

(T\_STACK) Contradictory—doesn't occur in source programs.

## Types $T$

- (WF\_BASE) Immediately true—WF\_BASE is an axiom.
- (WF\_REFINE) By the IH on  $e$  and WF\_REFINE.
- (WF\_FUN) By the IHs on  $T_1$  and  $T_2$  and WF\_FUN.

## Contexts $\Gamma$

- (WF\_EMPTY) Immediately true—WF\_EMPTY is an axiom.
- (WF\_EXTEND) By the IHs on  $\Gamma$  and  $T$  and WF\_EXTEND.

□

## B Proofs of space-efficiency soundness

This appendix contains the proofs relating classic  $\lambda_H$  to each other mode: forgetful, heedful, and eidetic.

### B.1 Relating classic and forgetful manifest contracts

If we evaluate a  $\lambda_H$  term with the classic semantics and find a value, then the forgetful semantics finds a similar value—identical if they’re constants. Since forgetful  $\lambda_H$  drops some casts, some terms reduce to blame in classic  $\lambda_H$  while they reduce to values in forgetful  $\lambda_H$ .

The relationship between classic and forgetful  $\lambda_H$  is *blame-inexact*, to borrow the terminology of Greenberg et al. [2012]: we define an asymmetric logical relation in Figure 12, relating classic values to forgetful values—and everything to classic blame. The proof proceeds largely like that of Greenberg et al. [2012]: we define a logical relation on terms and an inductive invariant relation on types, prove that casts between related types are logically related, and then show that well typed source programs are logically related.

Before we explain the logical relation proof itself, there is one new feature of the proof that merits discussion: we need to derive a congruence principle for casts forgetful  $\lambda_H$ . When proving that casts between related types are related (Lemma B.5), we want to be able to reason with the logical relation—which involves reducing the cast’s argument to a value. But if  $e \rightarrow_F^* e'$  such that  $\text{result}_F e'$ , how to  $\langle T_1 \dot{\Rightarrow} T_2 \rangle^l e$  and  $\langle T_1 \dot{\Rightarrow} T_2 \rangle^l e'$  relate? If  $e' = \uparrow l'$  is blame, then it may be that  $\langle T_1 \dot{\Rightarrow} T_2 \rangle^l e$  reduces to a value while  $\langle T_1 \dot{\Rightarrow} T_2 \rangle^l \uparrow l'$  propagates the blame. But if  $e'$  is a value, then both casts reduce to the same value. We show this property first for a single step  $e \rightarrow_F e'$ , and then lift it to many steps.

#### B.1 Lemma [Cast congruence (single step)]: If

- $\emptyset \vdash_F e : T_1$  and  $\emptyset \vdash_F \emptyset \parallel T_1 \Rightarrow T_2$  (and so  $\emptyset \vdash_F \langle T_1 \dot{\Rightarrow} T_2 \rangle^l e : T_2$ ),
- $e \rightarrow_F e_1$  (and so  $\emptyset \vdash_F e_1 : T_1$ ),
- $\langle T_1 \dot{\Rightarrow} T_2 \rangle^l e_1 \rightarrow_F^* e_2$ , and
- $\text{val}_F e_2$

then  $\langle T_1 \dot{\Rightarrow} T_2 \rangle^l e \rightarrow_F^* e_2$ .

**Proof:** By cases on the step  $e \rightarrow_F e_1$ . There are three groups of reductions: straightforward merge-free reductions, merging reductions (the interesting cases, where a reduction step taken in  $e$  has an exposed cast), and (contradictory) reductions where blame is raised.

#### Merge-free reductions

- (E\_BETA) By E\_CASTINNER and E\_BETA.
- (E\_OP) By E\_CASTINNER and E\_OP.
- (E\_UNWRAP) By E\_CASTINNER and E\_UNWRAP.
- (E\_APPL) By E\_CASTINNER with E\_APPL.
- (E\_APPR) By E\_CASTINNER with E\_APPR.
- (E\_CHECKOK) By E\_CASTINNER and E\_CHECKOK.
- (E\_OPINNER) By E\_CASTINNER and E\_OPINNER.

**Merging reductions** The interesting case of the proof occurs when the reduction step taken in  $e$  has an exposed cast: `E_CHECKNONE` or a congruence/merge rule (`E_CASTINNER` or `E_CASTMERGE`). Applying a cast to  $e$  and  $e_1$  leads to slightly different reductions, because we merge the cast in  $e$  and not in  $e_1$ . *If no blame is raised*, then the reductions join back up.

(`E_CHECKNONE`) We have  $e = (\langle T_3 \Rightarrow \{x:B \mid e_{11}\} \rangle^{l'} k)$  where  $T_1 = \{x:B \mid e_{11}\}$  and  $e_1 = \langle \{x:B \mid e_{11}\}, e_{11}[k/x], k \rangle^{l'}$  and  $\langle \{x:B \mid e_{11}\} \Rightarrow T_2 \rangle^l e_1 \rightarrow_F^* e_2$  such that  $\text{val}_F e_2$ . We must show that  $\langle \{x:B \mid e_{11}\} \Rightarrow T_2 \rangle^l e \rightarrow_F^* e_2$ .

By inversion of the similarity relation  $\vdash \{x:B \mid e_{11}\} \parallel T_2$ , we know that  $T_2 = \{x:B \mid e_{12}\}$ . If  $\langle \{x:B \mid e_{11}\} \Rightarrow \{x:B \mid e_{12}\} \rangle^l e_1$  reduces to a value, then it must be the case that  $e_1 = \langle \{x:B \mid e_{11}\}, e_{11}[k/x], k \rangle^{l'} \rightarrow_F^* k$  and that  $e_{12}[k/x] \rightarrow_F^* \text{true}$  (and so the entire term reduces  $\langle \{x:B \mid e_{11}\} \Rightarrow \{x:B \mid e_{12}\} \rangle^l e_1 \rightarrow_F^* k = e_2$ ). If not, we would have gotten  $\uparrow l'$  or  $\uparrow l$ .

Instead, we find that:

$$\begin{aligned} & \langle \{x:B \mid e_{11}\} \Rightarrow \{x:B \mid e_{12}\} \rangle^l (\langle T_3 \Rightarrow \{x:B \mid e_{11}\} \rangle^{l'} k) \\ \rightarrow_F & \langle T_3 \Rightarrow \{x:B \mid e_{12}\} \rangle^{l'} k \\ \rightarrow_F & \langle \{x:B \mid e_{12}\}, e_{12}[k/x], k \rangle^l \\ \rightarrow_F^* & \langle \{x:B \mid e_{12}\}, \text{true}, k \rangle^l \\ \rightarrow_F & k = e_2 \end{aligned}$$

(`E_CASTINNER`) We have:

$$e = \langle T_3 \Rightarrow T_1 \rangle^{l'} e_{11} \rightarrow_F \langle T_3 \Rightarrow T_1 \rangle^{l'} e_{12} = e_1$$

with  $e_{11} \rightarrow_F e_{12}$  and  $e_{11} \neq \langle T_4 \Rightarrow T_3 \rangle^{l''} e_2''$ .

In the original derivation with  $e_1$ , we have

$$\begin{aligned} & \langle T_1 \Rightarrow T_2 \rangle^l (\langle T_3 \Rightarrow T_1 \rangle^{l'} e_{12}) \rightarrow_F \langle T_3 \Rightarrow T_2 \rangle^l e_{12} \\ & \rightarrow_F^* e_2 \end{aligned}$$

by `E_CASTMERGE` and then assumption. We find a new derivation with  $e$  as follows:

$$\begin{aligned} & \langle T_1 \Rightarrow T_2 \rangle^l (\langle T_3 \Rightarrow T_1 \rangle^{l'} e_{11}) && (\text{E\_CASTMERGE}) \\ \rightarrow_F & \langle T_3 \Rightarrow T_2 \rangle^l e_{11} && (\text{E\_CASTINNER}) \\ & && \text{since } e_{11} \neq \langle T_4 \Rightarrow T_3 \rangle^{l''} e_2'' \\ \rightarrow_F & \langle T_3 \Rightarrow T_2 \rangle^l e_{12} && (\text{assumption}) \\ \rightarrow_F^* & e_2 \end{aligned}$$

(`E_CASTMERGE`) We have:

$$e = \langle T_3 \Rightarrow T_1 \rangle^{l'} (\langle T_4 \Rightarrow T_3 \rangle^{l''} e_{11}) \rightarrow_F \langle T_4 \Rightarrow T_1 \rangle^{l'} e_{11} = e_1$$

In the original derivation with  $e_1$ , we have

$$\langle T_1 \Rightarrow T_2 \rangle^l (\langle T_4 \Rightarrow T_1 \rangle^{l'} e_{11}) \rightarrow_F \langle T_4 \Rightarrow T_2 \rangle^l e_{11} \rightarrow_F^* e_2$$

We can build a new derivation with  $e$  as follows, stepping twice by `E_CASTMERGE`:

$$\begin{aligned} & \langle T_1 \Rightarrow T_2 \rangle^l (\langle T_3 \Rightarrow T_1 \rangle^{l'} (\langle T_4 \Rightarrow T_3 \rangle^{l''} e_{11})) \\ \rightarrow_F & \langle T_3 \Rightarrow T_2 \rangle^l (\langle T_4 \Rightarrow T_3 \rangle^{l''} e_{11}) \\ \rightarrow_F & \langle T_4 \Rightarrow T_2 \rangle^l e_{11} && (\text{assumption}) \\ \rightarrow_F^* & e_2 \end{aligned}$$

### Contradictory blame-raising reductions

(`E_APPRAISEL`) Contradiction—in this case,  $e_1 = \uparrow l'$ , and  $\langle T_1 \Rightarrow T_2 \rangle^l e_1 \rightarrow_F^* \uparrow l'$ , which isn't a value.

(`E_APPRAISER`) Contradiction—in this case,  $e_1 = \uparrow l'$ , and  $\langle T_1 \Rightarrow T_2 \rangle^l e_1 \rightarrow_F^* \uparrow l'$ , which isn't a value.

(`E_CASTRAISE`) Contradiction—in this case,  $e_1 = \uparrow l'$ , and  $\langle T_1 \Rightarrow T_2 \rangle^l e_1 \rightarrow_F^* \uparrow l'$ , which isn't a value.

(`E_CHECKFAIL`) Contradiction—in this case,  $e_1 = \uparrow l'$ , and  $\langle T_1 \Rightarrow T_2 \rangle^l e_1 \rightarrow_F^* \uparrow l'$ , which isn't a value.

(`E_OPRAISE`) Contradiction—in this case,  $e_1 = \uparrow l'$ , and  $\langle T_1 \Rightarrow T_2 \rangle^l e_1 \rightarrow_F^* \uparrow l'$ , which isn't a value.

(`E_CHECKRAISE`) Contradiction—in this case,  $e_1 = \uparrow l'$ , and  $\langle T_1 \Rightarrow T_2 \rangle^l e_1 \rightarrow_F^* \uparrow l'$ , which isn't a value.

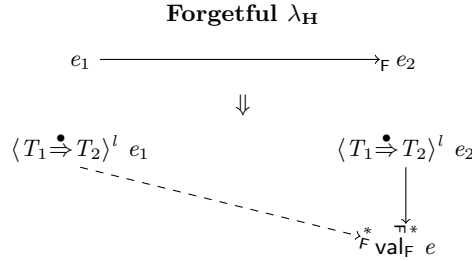
□

Once we have cast congruence for a single step, a straightforward induction gives us reasoning principle applicable to many steps.

**B.2 Lemma [Cast congruence]:** If

- $\emptyset \vdash_F e : T_1$  and  $\vdash_F \emptyset \parallel T_1 \Rightarrow T_2$  (and so  $\emptyset \vdash_F \langle T_1 \Rightarrow T_2 \rangle^l e : T_2$ ),
- $e \rightarrow_F^* e_1$  (and so  $\emptyset \vdash_F e_1 : T_1$ ),
- $\langle T_1 \Rightarrow T_2 \rangle^l e_1 \rightarrow_F^* e_2$ , and
- $\text{val}_F e_2$

then  $\langle T_1 \Rightarrow T_2 \rangle^l e \rightarrow_F^* e_2$ . Diagrammatically:



**Proof:** By induction on the derivation  $e \rightarrow_F^* e_1$ , using the single-step cast congruence (Lemma B.1). □

We define the logical relation in Figure 12. It is defined in a split style, with separate definitions for values and terms. Note that terms that classically reduce to blame are related to all forgetful terms, but terms that classically reduce to values reduce forgetfully to similar values. We lift these closed relations on values and terms to open terms by means of dual closing substitutions. As in Greenberg et al. [2012], we define an inductive invariant to relate types, using it to show that casts between related types on related values yield related values, i.e., casts are applicative (Lemma B.5). One important subtle technicality is that the type indices of this logical relation are forgetful types—in the constant case of the value relation, we evaluate the predicate in the forgetful semantics. We believe the choice is arbitrary, but have not tried the proof using classic type indices.

**B.3 Lemma [Value relation relates only values]:** If  $e_1 \sim_F e_2 : T$  then  $\text{val}_C e_1$  and  $\text{val}_F e_2$ .

**Proof:** By induction on  $T$ . We have  $e_1 = e_2 = k$  when  $T = \{x:B \mid e\}$  (and so we are done by V\_CONST). When  $T = T_1 \rightarrow T_2$ , we have the value derivations as assumptions. □

**B.4 Lemma [Relation implies similarity]:** If  $T_1 \sim_F T_2$  then  $\vdash T_1 \parallel T_2$ .

**Proof:** By induction on  $T_1$ , using S\_REFINE and S\_FUN. □

**B.5 Lemma [Relating classic and forgetful casts]:** If  $T_{11} \sim_F T_{21}$  and  $T_{12} \sim_F T_{22}$  and  $\vdash T_{11} \parallel T_{12}$ , then for all  $e_1 \sim_F e_2 : T_{21}$ , we have  $\langle T_{11} \Rightarrow T_{12} \rangle^l e_1 \simeq_F \langle T_{21} \Rightarrow T_{22} \rangle^l e_2 : T_{22}$ .

**Proof:** By induction on the sum of the heights of  $T_{21}$  and  $T_{22}$ . By Lemma B.4, we know that  $\vdash T_{11} \parallel T_{21}$  and  $\vdash T_{12} \parallel T_{22}$ ; by Lemma A.7, we know that  $\vdash T_{21} \parallel T_{22}$ . We go by cases on  $T_{22}$ .

( $T_{22} = \{x:B \mid e_{22}\}$ ) It must be the case (by similarity) that all of the other types are also refinements. Moreover, it must be that case that  $e_1 = e_2 = k$ .

Both sides step by E\_CHECKNONE. Since  $e_1 \sim_F e_2 : T_{21} = \{x:B \mid e_{21}\}$ , we can find that  $e_1 \sim_F e_2 : \{x:B \mid \text{true}\}$  trivially. Then, since  $\{x:B \mid e_{12}\} \sim_F \{x:B \mid e_{22}\}$ , we know that  $e_{12}[k/x] \simeq_F e_{22}[k/x] : \{x:\text{Bool} \mid \text{true}\}$ .

If  $e_{12}[k/x] \rightarrow_C^* \uparrow l'$ , then the entire classic side steps to  $\uparrow l'$  by E\_CHECKINNER and E\_CHECKRAISE, and then we are done. If not, then both predicates reduce to a boolean together. If they reduce to **false**, then the classic side eventually reduces to  $\uparrow l$  via E\_CHECKINNER and E\_CHECKFAIL, and we are done. If they both go to **true**, then both sides step by E\_CHECKINNER and E\_CHECKOK to yield  $k$ , and we can find  $k \sim_F k : \{x:B \mid e_{22}\}$  easily—we have a derivation for  $e_{22}[k/x] \rightarrow_F^* \text{true}$  handy.



( $T_{22} = T_{221} \rightarrow T_{222}$ ) By Lemma B.3, we know that  $\text{val}_C e_1$  and  $\text{val}_F e_2$ . The classic side is a value  $e_{11}$  (by  $V\_PROXYC$ ), while the forgetful side steps by one of its  $E\_CASTMERGE$  rules to some value  $e_{21}$ , depending on the shape of  $e_2$ : an abstraction yields a value by  $V\_PROXYF$ , while a function proxy yields another function proxy by  $E\_CASTMERGE$ .

We must now show that  $e_{11} \sim_F e_{21} : T_{221} \rightarrow T_{222}$ , knowing that  $e_1 \sim_F e_2 : T_{211} \rightarrow T_{212}$ . Let  $e_{12} \sim_F e_{22} : T_{221}$  be given. On the classic side, we step by  $E\_UNWRAP$  to find  $\langle T_{112} \Rightarrow T_{122} \rangle^l (e_1 (\langle T_{121} \Rightarrow T_{111} \rangle^l e_{12}))$ . (Recall that the annotations are all  $\bullet$ .)

We now go by cases on the step taken on the whether or not  $e_2$  is a value or needed to merge:

( $V\_PROXYF$ ) We have  $e_{21} = \langle T_{211} \rightarrow T_{212} \Rightarrow T_{221} \rightarrow T_{222} \rangle^l \lambda x : T_{211}. e'_2$  since  $e_2 = \lambda x : T_{211}. e'_2$ . We must show that:

$$\langle T_{112} \Rightarrow T_{122} \rangle^l (e_1 (\langle T_{121} \Rightarrow T_{111} \rangle^l e_{12})) \simeq_F (\langle T_{211} \rightarrow T_{212} \Rightarrow T_{221} \rightarrow T_{222} \rangle^l \lambda x : T_{211}. e'_2) e_{22} : T_{222}$$

The forgetful side steps by  $E\_UNWRAP$ , yielding  $\langle T_{212} \Rightarrow T_{222} \rangle^l ((\lambda x : T_{211}. e'_2) (\langle T_{221} \Rightarrow T_{211} \rangle^l e_{22}))$ . By the IH, we know that  $\langle T_{121} \Rightarrow T_{111} \rangle^l e_{12} \simeq_F \langle T_{221} \Rightarrow T_{211} \rangle^l e_{22} : T_{211}$ . If we get blame on the classic side, we are done immediately. Otherwise, each side reduces to values  $e'_{12} \sim_F e'_{22} : T_{211}$ . We know by assumption that  $e_1 e'_{12} \simeq_F e_2 e'_{22} : T_{212}$ ; again, blame on the classic side finishes this case. So suppose both sides go to values  $e''_{12} \sim_F e''_{22} : T_{212}$ . By the IH, we know that  $\langle T_{112} \Rightarrow T_{122} \rangle^l e'_{12} \simeq_F \langle T_{212} \Rightarrow T_{222} \rangle^l e''_{22} : T_{222}$ , and we are done.

( $E\_CASTMERGE$ ) We have  $e_{21} = \langle T_{31} \rightarrow T_{32} \Rightarrow T_{221} \rightarrow T_{222} \rangle^l \lambda x : T_{31}. e'_2$  since  $e_2 = \langle T_{31} \rightarrow T_{32} \Rightarrow T_{221} \rightarrow T_{212} \rangle^{l'} \lambda x : T_{31}. e'_2$ . We must show that:

$$\langle T_{112} \Rightarrow T_{122} \rangle^l (e_1 (\langle T_{121} \Rightarrow T_{111} \rangle^l e_{12})) \simeq_F (\langle T_{31} \rightarrow T_{32} \Rightarrow T_{221} \rightarrow T_{222} \rangle^l \lambda x : T_{31}. e'_2) e_{22} : T_{222}$$

The right hand steps by  $E\_UNWRAP$ , yielding  $\langle T_{32} \Rightarrow T_{222} \rangle^l ((\lambda x : T_{31}. e'_2) (\langle T_{221} \Rightarrow T_{31} \rangle^l e_{22}))$ . We must show that this forgetful term is related to the classic term  $\langle T_{112} \Rightarrow T_{122} \rangle^l (e_1 (\langle T_{121} \Rightarrow T_{111} \rangle^l e_{12}))$ .

We must now make a brief digression to examine the behavior of the cast that was eliminated by  $E\_CASTMERGE$ . We know by the IH that  $\langle T_{121} \Rightarrow T_{111} \rangle^l e_{12} \simeq_F \langle T_{221} \Rightarrow T_{211} \rangle^l e_{22} : T_{211}$ , so either the classic side goes to blame—and we are done—or both sides go to values  $e'_{12} \sim_F e'_{22} : T_{211}$ . By Lemma B.2, we can find that  $\langle T_{211} \Rightarrow T_{31} \rangle^l e'_{22} \rightarrow_F^* e''_{22}$  implies  $\langle T_{211} \Rightarrow T_{31} \rangle^l (\langle T_{221} \Rightarrow T_{211} \rangle^l e_{22}) \rightarrow_F^* e''_{22}$ . But then we have that  $\langle T_{211} \Rightarrow T_{31} \rangle^l (\langle T_{221} \Rightarrow T_{211} \rangle^l e_{22}) \rightarrow_F^* \langle T_{211} \Rightarrow T_{31} \rangle^l e_{22}$ , so we then know that  $\langle T_{221} \Rightarrow T_{31} \rangle^l e_{22} \rightarrow_F^* e''_{22}$ , just as if it were applied to  $e'_{22}$ .

Now we can return to the meat of our proof. If  $\langle T_{121} \Rightarrow T_{111} \rangle^l e_{12} \rightarrow_F^* \uparrow^{l'}$ , we are done. If it reduces to a value  $e'_{12}$ , then we are left considering the term  $\langle T_{112} \Rightarrow T_{122} \rangle^l (e_1 e'_{12})$  on the classic side. We know that  $e_1 \sim_F e_2 : T_{21}$ . Unfolding the definition of  $e_2$ , this means that  $e_1 e'_{12} \simeq_F \langle T_{32} \Rightarrow T_{212} \rangle^l ((\lambda x : T_{31}. e'_2) (\langle T_{211} \Rightarrow T_{31} \rangle^l e'_{22})) : T_{212}$ . If the classic side produces blame, we are done, as indicated in the digression above. If not, then both sides produce values. For these terms to produce values, it must be the case that (a) the domain cast on the forgetful side produces a value, (b) the forgetful function produces a value given that input, and (c) the forgetful codomain cast produces a value. Now, we know from our digression above that  $\langle T_{211} \Rightarrow T_{31} \rangle^l e'_{22}$  and  $\langle T_{211} \Rightarrow T_{31} \rangle^l e_{22}$  reduce to the exact same value,  $e''_{22}$ . So if  $\langle T_{32} \Rightarrow T_{212} \rangle^l ((\lambda x : T_{31}. e'_2) (\langle T_{211} \Rightarrow T_{31} \rangle^l e'_{22})) \rightarrow_F^* e''_{22}$  then we can also see

$$\langle T_{32} \Rightarrow T_{212} \rangle^l ((\lambda x : T_{31}. e'_2) (\langle T_{211} \Rightarrow T_{31} \rangle^l e_{22})) \rightarrow_F^* \langle T_{32} \Rightarrow T_{212} \rangle^l e_{32} \rightarrow_F^* e''_{22}.$$

We have shown that the domains and then the applied inner functions are equivalent. It now remains to show that

$$\langle T_{112} \Rightarrow T_{122} \rangle^l e'_{11} \simeq_F \langle T_{32} \Rightarrow T_{222} \rangle^l ((\lambda x : T_{31}. e'_2) (\langle T_{221} \Rightarrow T_{31} \rangle^l e_{22})) : T_{222}$$

We write the *entire* forgetful term to highlight the fact that we cannot freely apply congruence, but must instead carefully apply cast congruence (Lemma B.2) as we go.

By the IH, we know that either  $\langle T_{112} \Rightarrow T_{122} \rangle^l e'_{11}$  goes to blame or it goes to a value along with  $\langle T_{212} \Rightarrow T_{222} \rangle^l e'_{22}$ . In the former case we are done; in the latter case, we already know that  $\langle T_{32} \Rightarrow T_{212} \rangle^l e_{32} \rightarrow_F^* e''_{22}$ , so we can apply cast congruence (Lemma B.2) to see that if  $\langle T_{212} \Rightarrow T_{222} \rangle^l e'_{22} \rightarrow_F^* e''_{22}$  then  $\langle T_{212} \Rightarrow T_{222} \rangle^l (\langle T_{32} \Rightarrow T_{212} \rangle^l e_{32}) \rightarrow_F^* e''_{22}$ . But we know that  $\langle T_{212} \Rightarrow T_{222} \rangle^l (\langle T_{32} \Rightarrow T_{212} \rangle^l e_{32}) \rightarrow_F \langle T_{32} \Rightarrow T_{222} \rangle^l e_{32}$ , so we then we know that  $\langle T_{32} \Rightarrow T_{222} \rangle^l e_{32} \rightarrow_F^* e''_{22}$ . Since  $\langle T_{32} \Rightarrow T_{222} \rangle^l ((\lambda x : T_{31}. e'_2) (\langle T_{221} \Rightarrow T_{31} \rangle^l e_{22})) \rightarrow_F^* \langle T_{32} \Rightarrow T_{222} \rangle^l e_{32}$ , we have shown that the classic term and forgetful term reduce to values  $e'_{12} \sim_F e''_{22} : T_{222}$ , and we are done.  $\square$

## B.6 Lemma [Relating classic and forgetful source programs]:

1. If  $\Gamma \vdash_C e : T$  as a source program then  $\Gamma \vdash e \simeq_F e : T$ .

2. If  $\vdash_C T$  as a source program then  $T \sim_F T$ .

**Proof:** By mutual induction on the typing derivations.

### Term typing $\boxed{\Gamma \vdash_C e : T}$

**T\_VAR** We know by assumption that  $\delta_1(x) \sim_F \delta_2(x) : T$ .

**T\_CONST** Since we are dealing with a source program,  $T = \{x:B \mid \text{true}\}$ . We have immediately that  $\text{ty}(k) = B$  and  $\text{true}[k/x] \simeq_F \text{true}[k/x] : \{x:\text{Bool} \mid \text{true}\}$ , so  $k \simeq_F k : \{x:B \mid \text{true}\}$ .

**T\_ABS** Let  $\Gamma \models_F \delta$ . We must show that  $\lambda x:T_1. \delta_1(e_1) \sim_F \lambda x:T_2. \delta_2(e_1) : T_1 \rightarrow T_2$ . Let  $e_2 \sim_F e'_2 : T_1$ . We must show that applying the abstractions to these values yields related values. Both sides step by **E\_BETA**, to  $\delta_1(e_1)[e_2/x]$  and  $\delta_2(e_1)[e'_2/x]$ , respectively. But  $\Gamma, x:T_1 \models_F \delta[e_2, e'_2/x]$ , so we can apply IH (1)  $e_1$ , the two sides reduce to related values.

**T\_OP** By IH (1) on each arguments, either one of the arguments goes to blame in the classic evaluation, and we are done by **E\_OPRAISE**. Otherwise, all of the arguments reduce to related values. Since  $\text{ty}(op)$  is first order, these values must be related at refined base types, which means that they are in fact all equal constants. We then reduce by **E\_OP** on both sides to have  $\llbracket op \rrbracket(k_1, \dots, k_n)$ . We have assumed that the denotations of operations agree with their typings in *all* modes, so then  $\llbracket op \rrbracket(k_1, \dots, k_n)$  satisfies the refinement for  $\rightarrow_F$  in particular, and we are done.

**T\_APP** Let  $\Gamma \models_F \delta$ . We must show that  $\delta_1(e_1) \delta_2(e_2) \simeq_F \delta_2(e_1) \delta_2(e_2) : T_2$ . But by IH (1) on  $e_1$  and  $e_2$ , we are done directly.

**T\_CAST** Let  $\Gamma \models_F \delta$ . By IH (1) on  $e'$ ,  $\delta_1(e') \simeq_F \delta_2(e') : T_1$ , either  $\delta_1(e') \rightarrow_C^* \uparrow l'$  (and we are done) or  $\delta_1(e')$  and  $\delta_2(e')$  reduce to values  $e_1 \sim_F e_2 : T_1$ . By Lemma B.5 (using IH (2) on the types), we know that  $\langle T_1 \xrightarrow{\bullet} T_2 \rangle^l e_1 \simeq_F \langle T_1 \xrightarrow{\bullet} T_2 \rangle^l e_2 : T_2$ . If  $\langle T_1 \xrightarrow{\bullet} T_2 \rangle^l e_1 \rightarrow_C^* \uparrow l'$ , we are done. If not, then we know that the cast applied to both *values* reduce to values  $e'_1 \sim_F e'_2 : T_2$ , but we must still show that  $\langle T_1 \xrightarrow{\bullet} T_2 \rangle^l \delta_1(e') \simeq_F \langle T_1 \xrightarrow{\bullet} T_2 \rangle^l \delta_2(e') : T_2$  for the *terms*. The classic side obviously goes to  $e'_1$ . On the forgetful side, we can see by Lemma B.2 that  $\langle T_1 \xrightarrow{\bullet} T_2 \rangle^l e_2 \rightarrow_F^* e'_2$  implies  $\langle T_1 \xrightarrow{\bullet} T_2 \rangle^l \delta_2(e') \rightarrow_F^* e'_2$ , since  $\delta_2(e') \rightarrow_F^* e_2$ . Constructing this derivation completes the case.

**T\_BLAZE** Contradiction—doesn't appear in source programs. Though in fact it is in the relation, since  $\uparrow l \simeq_F e : T$  for any  $e$  and  $T$ .

**T\_CHECK** Contradiction—doesn't appear in source programs.

### Type well formedness $\boxed{\vdash_C T}$

**WF\_BASE** We can immediately see  $\text{true}[k/x] \simeq_F \text{true}[k/x] : \{x:\text{Bool} \mid \text{true}\}$  for any  $k \sim_F k : \{x:B \mid \text{true}\}$ , i.e., any  $k$  such that  $\text{ty}(k) = B$ .

**WF\_REFINE** By inversion, we know that  $x:\{x:B \mid \text{true}\} \vdash_C e : \{x:\text{Bool} \mid \text{true}\}$ ; by IH (1), we find that  $\delta_1(e) \simeq_F \delta_2(e) : \{x:\text{Bool} \mid \text{true}\}$ , i.e., that  $e[e_1/x] \simeq_F e[e_2/x] : \{x:\text{Bool} \mid \text{true}\}$  for all  $e_1 \sim_F e_2 : \{x:B \mid \text{true}\}$ —which is what we needed to know.

**WF\_FUN** By IH (2) on each of the types.

□

## B.2 Relating classic and heedful manifest contracts

Heedful  $\lambda_H$  reorders casts, so we won't necessarily get the same blame as we do in classic  $\lambda_H$ . We can show, however, that they blame the same amount: heedful  $\lambda_H$  raises blame if and only if classic  $\lambda_H$  does, too. We define a blame-inexact, symmetric logical relation.

The proof follows the same scheme as the proof for forgetful  $\lambda_H$  in Section B.1: we first prove a cast congruence principle; then we define a logical relation relating classic and heedful  $\lambda_H$ ; we prove a lemma establishing a notion of applicativity for casts using an inductive invariant grounded in the logical relation, and then use that lemma to prove that well typed source programs are logically related.

Cast congruence—that  $\langle T_1 \xrightarrow{\bullet} T_2 \rangle^l e$  and  $\langle T_1 \xrightarrow{\bullet} T_2 \rangle^l e_1$  behave identically when  $e \rightarrow_H e_1$ —holds almost exactly. The pre- and post-step terms may end blaming different labels, but otherwise return identical values. Note that this cast congruence lemma (a) has annotations other than  $\bullet$ , and (b) is stronger than Lemma B.1, since we not only get the same value out, but we also get blame when the inner reduction yields blame—though the label may be different. The potentially different blame labels in heedful  $\lambda_H$ 's cast congruence principle arises because of how casts are merged: heedful  $\lambda_H$  is heedful of types, but forgets blame labels.

**B.7 Lemma [First-order casts don't change their arguments]:** If  $\langle \{x:B \mid e_1\} \xrightarrow{S} \{x:B \mid e_2\} \rangle^l k \rightarrow_{\mathbb{H}}^* e$  and  $\text{val}_{\mathbb{H}} e$  then  $e = k$ .

**Proof:** By induction on the size of  $\mathcal{S}$ .

( $\mathcal{S} = \emptyset$ ) The only possible step is  $\langle \{x:B \mid e_1\} \xrightarrow{\emptyset} \{x:B \mid e_2\} \rangle^l k \rightarrow_{\mathbb{H}} \langle \{x:B \mid e_2\}, e_2[k/x], k \rangle^l$  by `E-CHECKEMPTY`. So if the original cast term reduces to a value, then so must this term. But the only step out of an active check that produces a value produces  $k$  by `E-CHECKOK`.

( $\mathcal{S} = \{\{x:B \mid e_3\}\} \cup \mathcal{S}'$ ) If  $\mathcal{S}$  is nonempty, then we must step by `E-CHECKSET` for some  $\{x:B \mid e_3\} \in \mathcal{S}$  to  $\langle \{x:B \mid e_3\} \xrightarrow{\mathcal{S} \setminus \{\{x:B \mid e_3\}\}} \{x:B \mid e_2\} \rangle^l \langle \{x:B \mid e_3\}, e_3[k/x], k \rangle^l$ . For this entire term to reduce to a value, the active check must reduce to a value—if it goes to blame, so does the whole term. But the only value it can produce is  $k$  itself, by `E-CHECKOK`. By the IH, we know that  $\langle \{x:B \mid e_3\} \xrightarrow{\mathcal{S} \setminus \{\{x:B \mid e_3\}\}} \{x:B \mid e_2\} \rangle^l k$  goes to  $k$  if it reduces to a value.  $\square$

**B.8 Lemma [Determinism of heedful  $\lambda_{\mathbb{H}}$ ]:** If  $e \rightarrow_{\mathbb{H}} e_1$  and  $e \rightarrow_{\mathbb{H}} e_2$  then  $e_1 = e_2$ .

**Proof:** By induction on the first evaluation derivation. In every case, only a single step can be taken. Critically, `E-CHECKSET` uses the `choose` function, which makes some deterministic choice.  $\square$

Heedful  $\lambda_{\mathbb{H}}$ 's cast congruence proof requires an extra principle. We first show that casting is idempotent: we can safely remove the source type from a type set.

**B.9 Lemma [Idempotence of casts]:**

If  $\emptyset \vdash_{\mathbb{H}} \langle \{x:B \mid e_1\} \xrightarrow{S} \{x:B \mid e_2\} \rangle^l k : \{x:B \mid e_2\}$  and  $\emptyset \vdash_{\mathbb{H}} k : \{x:B \mid e_3\}$  then for all  $\text{result}_{\mathbb{H}} e$ , then:

- (a)  $\langle \{x:B \mid e_1\} \xrightarrow{S} \{x:B \mid e_2\} \rangle^l k \rightarrow_{\mathbb{H}}^* e$  iff
- (b)  $\langle \{x:B \mid e_1\} \xrightarrow{\mathcal{S} \setminus \{\{x:B \mid e_3\}\}} \{x:B \mid e_2\} \rangle^l k \rightarrow_{\mathbb{H}}^* e$ .

**Proof:** If  $\{x:B \mid e_1\} \notin \mathcal{S}$ , then the proof is trivial, since the (a) and (b) are the same. The rest of the proof assumes that  $\{x:B \mid e_1\} \in \mathcal{S}$ .

We prove both directions by induction on  $\mathcal{S}$ . In both cases,  $\mathcal{S} = \emptyset$  is immediate—the two are the same!

For the only if ( $\Leftarrow$ ) direction, if `choose`( $\mathcal{S}$ ) =  $\{x:B \mid e_1\}$ , then we step (a) by `E-CHECKSET` choosing  $\{x:B \mid e_1\}$ , finding  $\langle \{x:B \mid e_1\} \xrightarrow{\mathcal{S} \setminus \{\{x:B \mid e_1\}\}} \{x:B \mid e_2\} \rangle^l \langle \{x:B \mid e_1\}, e_1[k/x], k \rangle^l$ . By inversion of the typing derivation,  $e_1[k/x] \rightarrow_{\mathbb{H}}^* \text{true}$ , so we can step by `E-CHECKINNER` and `E-CHECKOK` to  $\langle \{x:B \mid e_1\} \xrightarrow{\mathcal{S} \setminus \{\{x:B \mid e_1\}\}} \{x:B \mid e_2\} \rangle^l k$ , which is exactly (b), and we are done. If not, the two casts step to the same sub-checks and co-reduce until eventually `choose`( $\mathcal{S}$ ) =  $\{x:B \mid e_1\}$ .

For the if ( $\Rightarrow$ ) direction, (a) chooses a type to check and steps by `E-CHECKSET` to  $\langle \{x:B \mid e_3\} \xrightarrow{\mathcal{S} \setminus \{\{x:B \mid e_3\}\}} \{x:B \mid e_2\} \rangle^l \langle \{x:B \mid e_3\}, e_3[k/x], k \rangle^l$  where `choose`( $\mathcal{S}$ ) =  $\{x:B \mid e_3\}$ . If  $e_3 = e_1$ , then we know by the typing derivation that  $e_3[k/x] \rightarrow_{\mathbb{H}}^* \text{true}$ , so the active check must succeed and (a) necessarily steps to  $\langle \{x:B \mid e_1\} \xrightarrow{\mathcal{S} \setminus \{\{x:B \mid e_1\}\}} \{x:B \mid e_2\} \rangle^l k$  by determinism (Lemma B.8)—and we are done, since this is the term for which we needed to produce a derivation.

If  $e_3 \neq e_1$ , then we take a similar step in (b) to  $\langle \{x:B \mid e_3\} \xrightarrow{\mathcal{S} \setminus \{\{x:B \mid e_1\}\} \setminus \{\{x:B \mid e_3\}\}} \{x:B \mid e_2\} \rangle^l \langle \{x:B \mid e_3\}, e_3[k/x], k \rangle^l$ . Now whatever the derivation for (a) does to the active check, we can recapitulate in (b). If (a) produces  $\uparrow l'$  for some  $l'$ , via either `E-CHECKRAISE` or `E-CHECKFAIL`, then we are done with the whole proof. If (a) produces a value, it must produce  $k$  itself by `E-CHECKOK`. But by the IH we know that  $\langle \{x:B \mid e_3\} \xrightarrow{\mathcal{S} \setminus \{\{x:B \mid e_3\}\}} \{x:B \mid e_2\} \rangle^l k \rightarrow_{\mathbb{H}}^* e$  implies that  $\langle \{x:B \mid e_3\} \xrightarrow{\mathcal{S} \setminus \{\{x:B \mid e_3\}\} \setminus \{\{x:B \mid e_1\}\}} \{x:B \mid e_2\} \rangle^l k \rightarrow_{\mathbb{H}}^* e$ , and we are done.  $\square$

We need strong normalization to prove cast congruence: if we reorder checks, we need to know that reordering checks doesn't change the observable behavior. We define a unary logical relation to show strong normalization in Figure 13. We assume throughout at the terms are well typed at their indices:  $e \in \llbracket T \rrbracket$  implies  $\emptyset \vdash_{\mathbb{H}} e : T$  and  $\models T$  implies  $\vdash_{\mathbb{H}} T$  and  $\models \mathcal{S} \parallel T_1 \Rightarrow T_2$  implies  $\vdash_{\mathbb{H}} \mathcal{S} \parallel T_1 \Rightarrow T_2$  and  $\Gamma \models e : T$  implies  $\Gamma \vdash_{\mathbb{H}} e : T$  by definition. Making this assumption simplifies many of the technicalities. First, typed terms stay well typed as they evaluate (by preservation, Lemma A.26), so a well typed relation allows us to reason exclusively over typed terms. Second, it allows us to ignore the refinements in our relation, essentially using the simple type structure. After proving cast congruence, we show that all well typed terms are in fact in the relation, i.e., that all heedful terms normalize.

**B.10 Lemma [Expansion and contraction]:** If  $e_1 \rightarrow_{\mathbb{H}}^* e_2$  then  $e_1 \in \llbracket T \rrbracket$  iff  $e_2 \in \llbracket T \rrbracket$ .

Normalizing closed terms

$$e \in \llbracket T \rrbracket$$

$$\begin{aligned} e \in \llbracket \{x:B \mid e\} \rrbracket &\iff e \xrightarrow{*}_{\mathsf{H}} \uparrow l \vee \\ &\quad e \xrightarrow{*}_{\mathsf{H}} k \wedge \mathsf{ty}(k) = B \\ e \in \llbracket T_1 \rightarrow T_2 \rrbracket &\iff \forall e' \in \llbracket T_1 \rrbracket. \mathsf{result}_{\mathsf{H}} e' \Rightarrow e e' \in \llbracket T_2 \rrbracket \end{aligned}$$

Normalizing open terms

$$\Gamma \models e : T \quad \Gamma \models \sigma$$

$$\begin{aligned} \Gamma \models e : T &\iff \forall \sigma. \Gamma \models \sigma \Rightarrow \sigma(e) \in \llbracket T \rrbracket \\ \Gamma \models \sigma &\iff \forall x:T \in \Gamma. \sigma(x) \in \llbracket T \rrbracket \end{aligned}$$

Normalizing types and type sets

$$\models T \quad \models \mathcal{S} \parallel T_1 \Rightarrow T_2$$

$$\begin{aligned} &\frac{\forall k. \mathsf{ty}(k) = B \text{ implies } e[k/x] \in \llbracket \{x:\mathsf{Bool} \mid \mathsf{true}\} \rrbracket}{\models \{x:B \mid e\}} \quad \mathsf{SWF\_REFINE} \\ &\frac{\models T_1 \quad \models T_2}{\models T_1 \rightarrow T_2} \quad \mathsf{SWF\_FUN} \\ &\frac{\vdash T_1 \parallel T_2 \quad \models T_1 \quad \models T_2 \quad \forall T \in \mathcal{S}. \models T \quad \vdash T \parallel T_1}{\models \mathcal{S} \parallel T_1 \Rightarrow T_2} \quad \mathsf{SWF\_TYPESET} \end{aligned}$$

Figure 13: Strong normalization for heedful  $\lambda_{\mathsf{H}}$

**Proof:** By induction on  $T$ .

( $T = \{x:B \mid e''\}$ ) By determinism (Lemma B.8).

( $T = T_1 \rightarrow T_2$ ) Given some  $e' \in \llbracket T_1 \rrbracket$ , we must show that  $e_1 e' \in \llbracket T_2 \rrbracket$  iff  $e_2 e' \in \llbracket T_2 \rrbracket$ . We have  $e_1 e' \xrightarrow{*}_{\mathsf{H}} e_2 e'$  by induction on the length of the evaluation derivation and  $\mathsf{E\_APPL}$ , so we are done by the IH on  $T_2$ .  $\square$

**B.11 Lemma [Blame inhabits every type]:**  $\uparrow l \in \llbracket T \rrbracket$  for all  $T$ .

**Proof:** By induction on  $T$ .

( $T = \{x:B \mid e''\}$ ) By definition.

( $T = T_1 \rightarrow T_2$ ) By the IH,  $\uparrow l' \in \llbracket T_1 \rrbracket$ . We must show that  $\uparrow l \uparrow l' \in \llbracket T_2 \rrbracket$ . This term steps to  $\uparrow l$  by  $\mathsf{E\_APPRAISEL}$ , and then we are done by contraction (Lemma B.10).  $\square$

**B.12 Lemma [Strong normalization]:** If  $e \in \llbracket T \rrbracket$  then  $e \xrightarrow{*}_{\mathsf{H}} e'$  uniquely such that  $\mathsf{result}_{\mathsf{H}} e'$ .

**Proof:** Uniqueness is immediate by determinism (Lemma B.8). We show normalization by induction on  $T$ , observing that blame inhabits every type.

( $T = \{x:B \mid e''\}$ ) By definition.

( $T = T_1 \rightarrow T_2$ ) By Lemma B.11, we know that at least one result is in the domain type:  $\uparrow l \in \llbracket T_1 \rrbracket$ . So by assumption,  $e \uparrow l \in \llbracket T_2 \rrbracket$ . By the IH, this term is strong normalizing—but that can only be so if  $e$  reduces to a result.  $\square$

**B.13 Lemma [Cast congruence (single step)]:** If

- $e \in \llbracket T_1 \rrbracket$  and  $\models \mathcal{S} \parallel T_1 \Rightarrow T_2$  (and so  $\emptyset \vdash_{\mathsf{H}} \langle T_1 \xrightarrow{\mathcal{S}} T_2 \rangle^l e : T_2$ ),
- $e \xrightarrow{*}_{\mathsf{H}} e_1$  (and so  $\emptyset \vdash_{\mathsf{H}} e_1 : T_1$ ),
- $\langle T_1 \xrightarrow{\mathcal{S}} T_2 \rangle^l e_1 \xrightarrow{*}_{\mathsf{H}} e_2$ , and
- $\mathsf{result}_{\mathsf{H}} e_2$

then  $\langle T_1 \xrightarrow{\mathcal{S}} T_2 \rangle^l e \xrightarrow{*}_{\mathsf{H}} \uparrow l'$  if  $e_2 = \uparrow l$  or to  $e_2$  itself if  $\mathsf{val}_{\mathsf{H}} e_2$ .

**Proof:** By cases on the step taken; the proof is as for forgetful  $\lambda_{\mathsf{H}}$  (Lemma B.1), though we need to use strong normalization to handle the reorderings. There are two groups of reductions: straightforward merge-free reductions and merging reductions.

**Merge-free reductions** In these cases, we apply  $\text{E\_CASTINNER}$  and whatever rule derived  $e \rightarrow_{\text{H}} e_1$ .

- ( $\text{E\_BETA}$ ) By  $\text{E\_CASTINNER}$  and  $\text{E\_BETA}$ .
- ( $\text{E\_OP}$ ) By  $\text{E\_CASTINNER}$  and  $\text{E\_OP}$ .
- ( $\text{E\_UNWRAP}$ ) By  $\text{E\_CASTINNER}$  and  $\text{E\_UNWRAP}$ .
- ( $\text{E\_TYPESET}$ ) By  $\text{E\_CASTINNER}$  and  $\text{E\_TYPESET}$ .
- ( $\text{E\_APPL}$ ) By  $\text{E\_CASTINNER}$  with  $\text{E\_APPL}$ .
- ( $\text{E\_APPR}$ ) By  $\text{E\_CASTINNER}$  with  $\text{E\_APPR}$ .
- ( $\text{E\_APPRRAISEL}$ ) By  $\text{E\_CASTINNER}$  with  $\text{E\_APPRRAISEL}$ ; then by  $\text{E\_CASTRAISE}$  on both sides.
- ( $\text{E\_APPRRAISER}$ ) By  $\text{E\_CASTINNER}$  with  $\text{E\_APPRRAISER}$ ; then by  $\text{E\_CASTRAISE}$  on both sides.
- ( $\text{E\_CHECKOK}$ ) By  $\text{E\_CASTINNER}$  and  $\text{E\_CHECKOK}$ .
- ( $\text{E\_CHECKFAIL}$ ) By  $\text{E\_CASTINNER}$  and  $\text{E\_CHECKRAISE}$ .
- ( $\text{E\_CHECKFAIL}$ ) By  $\text{E\_CASTINNER}$  and  $\text{E\_CHECKRAISE}$ .
- ( $\text{E\_OPINNER}$ ) By  $\text{E\_CASTINNER}$  and  $\text{E\_OPINNER}$ .
- ( $\text{E\_OPRAISE}$ ) By  $\text{E\_CASTINNER}$  and  $\text{E\_OPRAISE}$ .

**Merging reductions** In these cases, some cast in  $e$  reduces when we step  $e \rightarrow_{\text{H}} e_1$ , but merges when we consider  $\langle T_1 \xrightarrow{\mathcal{S}} T_2 \rangle^l e$ . We must show that the merged term and  $\langle T_1 \xrightarrow{\mathcal{S}} T_2 \rangle^l e_1$  eventually meet. After merging the cast in  $e$ —and possibly some steps in  $e_1$ —the  $e$  and  $e_1$  terms reduce to a common term, which immediately gives us the common reduction to results we need.

( $\text{E\_CHECKEMPTY}$ ) We have  $e = (\langle T_3 \xrightarrow{\mathcal{Q}} \{x:B \mid e_{11}\} \rangle^{l'} k)$  where  $T_1 = \{x:B \mid e_{11}\}$  and  $e_1 = \langle \{x:B \mid e_{11}\}, e_{11}[k/x], k \rangle^{l'}$  and  $\langle \{x:B \mid e_{11}\} \xrightarrow{\mathcal{S}} T_2 \rangle^l e_1 \rightarrow_{\text{H}}^* e_2$  such that  $\text{result}_{\text{H}} e_2$ . We must show that  $\langle \{x:B \mid e_{11}\} \xrightarrow{\mathcal{S}} T_2 \rangle^l e \rightarrow_{\text{H}}^* e'_2$  such that  $e_2 = \uparrow l$  and  $e'_2 = \uparrow l'$  or  $\text{val}_{\text{H}} e_2$  and  $e_2 = e'_2$ . We find that both terms go to blame (at possibly different labels), or reduce to  $\langle \{x:B \mid e_{11}\} \xrightarrow{\mathcal{S}} T_2 \rangle^l k$ .

We step the  $e$  term:

$$\begin{aligned}
& \langle \{x:B \mid e_{11}\} \xrightarrow{\mathcal{S}} T_2 \rangle^l (\langle T_3 \xrightarrow{\bullet} \{x:B \mid e_{11}\} \rangle^{l'} k) && \text{E\_CASTMERGE} \\
\rightarrow_{\text{H}} & \langle T_3 \xrightarrow{\mathcal{S} \cup \{\{x:B \mid e_{11}\}\}} T_2 \rangle^l k && \text{E\_CHECKSET} \\
\rightarrow_{\text{H}} & \langle \{x:B \mid e_{11}\} \xrightarrow{\mathcal{S} \setminus \{x:B \mid e_{11}\}} T_2 \rangle^l \langle \{x:B \mid e_{11}\}, e_{11}[k/x], k \rangle^l
\end{aligned}$$

Knowing that  $\langle \{x:B \mid e_{11}\} \xrightarrow{\mathcal{S}} T_2 \rangle^l \langle \{x:B \mid e_{11}\}, e_{11}[k/x], k \rangle^{l'} \rightarrow_{\text{H}}^* e_2$ , we know that  $e_{11}[k/x] \rightarrow_{\text{H}}^* e'_{11}$  such that  $\text{result}_{\text{H}} e'_{11}$ . If it goes to  $\uparrow l''$ , so do both the  $e_1$  and  $e$  terms by  $\text{E\_CHECKRAISE}$ . If it goes to **false**, the  $e_1$  term goes to  $\uparrow l'$  while the  $e$  term goes to  $\uparrow l$ , both by  $\text{E\_CHECKFAIL}$ . Finally, if it goes to **true**, then we know that  $\langle \{x:B \mid e_{11}\} \xrightarrow{\mathcal{S}} T_2 \rangle^l e_1 = \langle \{x:B \mid e_{11}\} \xrightarrow{\mathcal{S}} T_2 \rangle^l \langle \{x:B \mid e_{11}\}, e_{11}[k/x], k \rangle^{l'} \rightarrow_{\text{H}}^* \langle T_1 \xrightarrow{\mathcal{S}} T_2 \rangle^l k$ . If  $\{x:B \mid e_{11}\} \notin \mathcal{S}$ , then we are already done—we say that the  $e$  term stepped to this.

If, on the other hand,  $\{x:B \mid e_{11}\} \in \mathcal{S}$ , then the  $e$  term stepped to  $\langle \{x:B \mid e_{11}\} \xrightarrow{\mathcal{S} \setminus \{x:B \mid e_{11}\}} T_2 \rangle^l k$  while the  $e_1$  term stepped to  $\langle \{x:B \mid e_{11}\} \xrightarrow{\mathcal{S}} T_2 \rangle^l k$ . We can apply reflexivity of casts (Lemma B.9) to see that these terms reduce to the same results.

( $\text{E\_CHECKSET}$ ) This case is quite similar to  $\text{E\_CHECKEMPTY}$ . We have  $e = (\langle T_3 \xrightarrow{\mathcal{S}_2} T_1 \rangle^{l'} k)$  where  $\text{choose}(\mathcal{S}_2) = \{x:B \mid e_{11}\}$  and  $e_1 = \langle \{x:B \mid e_{11}\} \xrightarrow{\mathcal{S}_2 \setminus \{x:B \mid e_{11}\}} T_1 \rangle^{l'} \langle \{x:B \mid e_{11}\}, e_{11}[k/x], k \rangle^{l'}$  and  $\langle \{x:B \mid e_{11}\} \xrightarrow{\mathcal{S}} T_2 \rangle^l e_1 \rightarrow_{\text{H}}^* e_2$  such that  $\text{result}_{\text{H}} e_2$ . We find that both sides reduce to blame (at possibly different labels) or the common term  $\langle \{x:B \mid e_{11}\} \xrightarrow{(\mathcal{S}_1 \cup \mathcal{S}_2 \cup \{T_1\}) \setminus \{x:B \mid e_{11}\}} T_2 \rangle^l k$ .

We step the  $e$  term:

$$\begin{aligned}
& \langle T_1 \xrightarrow{\mathcal{S}_1} T_2 \rangle^l (\langle T_3 \xrightarrow{\mathcal{S}_2} T_1 \rangle^{l'} k) && \text{E\_CASTMERGE} \\
\rightarrow_{\text{H}} & \langle T_3 \xrightarrow{\mathcal{S}_1 \cup \mathcal{S}_2 \cup \{T_1\}} T_2 \rangle^l k && \text{E\_CHECKSET}
\end{aligned}$$

Similarly, we know that the  $e_1$  term must step by  $\text{E\_CASTMERGE}$  as well:

$$\begin{aligned}
& \langle T_1 \xrightarrow{\mathcal{S}_1} T_2 \rangle^l (\langle \{x:B \mid e_{11}\} \xrightarrow{\mathcal{S}_2 \setminus \{x:B \mid e_{11}\}} T_1 \rangle^{l'} \langle \{x:B \mid e_{11}\}, e_{11}[k/x], k \rangle^{l'}) \rightarrow_{\text{H}} \\
& \langle \{x:B \mid e_{11}\} \xrightarrow{(\mathcal{S}_1 \cup \mathcal{S}_2 \cup \{T_1\}) \setminus \{x:B \mid e_{11}\}} T_2 \rangle^l \langle \{x:B \mid e_{11}\}, e_{11}[k/x], k \rangle^{l'}
\end{aligned}$$

Knowing that  $\langle \{x:B \mid e_{11}\} \xrightarrow{\mathcal{S}_2 \setminus \{x:B \mid e_{11}\}} T_2 \rangle^l \langle \{x:B \mid e_{11}\}, e_{11}[k/x], k \rangle^{l'} \rightarrow_{\text{H}}^* e_2$ , we know that  $e_{11}[k/x] \rightarrow_{\text{H}}^* e'_{11}$  such that  $\text{result}_{\text{H}} e'_{11}$ . If it goes to  $\uparrow l''$ , so does the  $e_1$  and by  $\text{E\_CHECKRAISE}$  followed by  $\text{E\_CASTRAISE}$ . The  $e$

term, depending on what **choose** selects, produces either a different blame label (because the types it checks first fail) or it eventually chooses  $\{x:B \mid e_{11}\}$  and raises blame, too. Note that here we are relying critically on strong normalization, Lemma B.12 and WF\_TYPESET, to see that all checks reduce to results on both sides. A similar case adheres when the check goes to **false**. Finally, if it goes to **true**, then we know that the  $e_1$  term reduces  $\langle \{x:B \mid e_{11}\}^{(S_1 \cup S_2 \cup \{T_1\}) \setminus \{x:B \mid e_{11}\}} T_2 \rangle^l \langle \{x:B \mid e_{11}\}, e_{11}[k/x], k \rangle^{l'} \rightarrow_H^* \langle \{x:B \mid e_{11}\}^{(S_1 \cup S_2 \cup \{T_1\}) \setminus \{x:B \mid e_{11}\}} T_2 \rangle^l k$  on its way to the result  $e_2$ . We can then reduce the two terms together as new types are chosen (from  $S_1 \cup S_2 \cup \{T_1\} \setminus \{x:B \mid e_{11}\}$  and from  $S_1 \cup S_2 \cup \{T_1\}$ ) until  $\{x:B \mid e_{11}\}$  is eliminated from the  $e$ -term's type set, and the two terms are the same.

(E\_CASTINNER) We have  $e = \langle T_3 \xrightarrow{S_2} T_1 \rangle^{l'} e_{11} \rightarrow_H \langle T_3 \xrightarrow{S_2} T_1 \rangle^{l'} e_{12} = e_1$ , with  $e_{11} \rightarrow_H e_{12}$  and  $e_{11} \neq \langle T_4 \xrightarrow{S_1} T_3 \rangle^{l''} e_2''$ . We reduce both to the common term  $\langle T_3^{S_1 \cup S_2 \cup \{T_1\}} T_2 \rangle^l e_{12}$ .

In the original derivation with  $e_1$ , the only step we can take is  $\langle T_1 \xrightarrow{S_1} T_2 \rangle^l (\langle T_3 \xrightarrow{S_2} T_1 \rangle^{l'} e_{12}) \rightarrow_H \langle T_3^{S_1 \cup S_2 \cup \{T_1\}} T_2 \rangle^l e_{12}$  E\_CASTMERGE. We find a new derivation with  $e$  as follows:

$$\begin{array}{ccc} \langle T_1 \xrightarrow{S_1} T_2 \rangle^l (\langle T_3 \xrightarrow{S_2} T_1 \rangle^{l'} e_{11}) & & \text{E\_CASTMERGE} \\ \rightarrow_H \langle T_3^{S_1 \cup S_2 \cup \{T_1\}} T_2 \rangle^l e_{11} & \text{E\_CASTINNER since } e_{11} \neq \langle T_4 \xrightarrow{S_1} T_3 \rangle^{l''} e_2'' & \\ \rightarrow_H \langle T_3^{S_1 \cup S_2 \cup \{T_1\}} T_2 \rangle^l e_{12} & & \text{(assumption)} \end{array}$$

(E\_CASTMERGE) We have  $e = \langle T_3 \xrightarrow{S_2} T_1 \rangle^{l'} (\langle T_4 \xrightarrow{S_3} T_3 \rangle^{l''} e_{11}) \rightarrow_H \langle T_4^{S_2 \cup S_3 \cup \{T_3\}} T_1 \rangle^{l'} e_{11} = e_1$ .

In the original derivation with  $e_1$ , the only step we can take is  $\langle T_1 \xrightarrow{S_1} T_2 \rangle^l (\langle T_4^{S_2 \cup S_3 \cup \{T_3\}} T_1 \rangle^{l'} e_{11}) \rightarrow_H \langle T_4^{S_1 \cup S_2 \cup S_3 \cup \{T_1\} \cup \{T_3\}} T_2 \rangle^l e_{11}$ . We can build a new derivation with  $e$  as follows:

$$\begin{array}{ccc} \langle T_1 \xrightarrow{S_1} T_2 \rangle^l (\langle T_3 \xrightarrow{S_2} T_1 \rangle^{l'} (\langle T_4 \xrightarrow{S_3} T_3 \rangle^{l''} e_{11})) & & \text{E\_CASTMERGE} \\ \rightarrow_H \langle T_3^{S_1 \cup S_2 \cup \{T_1\}} T_2 \rangle^l (\langle T_4 \xrightarrow{S_3} T_3 \rangle^{l''} e_{11}) & \text{E\_CASTMERGE} & \\ \rightarrow_H \langle T_4^{S_1 \cup S_2 \cup S_3 \cup \{T_1\} \cup \{T_3\}} T_2 \rangle^l e_{11} & & \text{(assumption)} \end{array}$$

(E\_CASTRAISE) By E\_CASTMERGE, we can reduce  $e$  to  $\langle T_1 \xrightarrow{S_1} T_2 \rangle^l \uparrow l'$ , which is just the same term as  $\langle T_1 \xrightarrow{S_1} T_2 \rangle^l e_1$ . □

**B.14 Lemma [Cast congruence]:** If

- $\emptyset \models e : T_1$  and  $\models \mathcal{S} \parallel T_1 \Rightarrow T_2$  (and so  $\emptyset \vdash_H \langle T_1 \xrightarrow{S_1} T_2 \rangle^l e : T_2$ ),
- $e \rightarrow_H^* e_1$  (and so  $\emptyset \vdash_H e_1 : T_1$ ),
- $\langle T_1 \xrightarrow{S_1} T_2 \rangle^l e_1 \rightarrow_H^* e_2$ , and
- $\text{result}_H e_2$

then  $\langle T_1 \xrightarrow{S_1} T_2 \rangle^l e \rightarrow_H^* \uparrow l'$  if  $e_2 = \uparrow l'$  or to  $e_2$  itself if  $\text{val}_H e_2$ . Diagrammatically:

$$\begin{array}{ccc} & \text{Heedful } \lambda_H & \\ e_1 & \xrightarrow{\quad} e_2 & \\ & \Downarrow & \\ \langle T_1 \xrightarrow{S_1} T_2 \rangle^l e_1 & & \langle T_1 \xrightarrow{S_1} T_2 \rangle^l e_2 \\ \vdots & & \downarrow \\ \text{result}_H^* e_1' & \sim & \text{result}_H^* e_2' \\ & \text{val}_H e & \end{array}$$

**Proof:** By induction on the derivation  $e \rightarrow_H^* e_1$ , using the single-step cast congruence (Lemma B.13). □

**B.15 Lemma [Strong normalization of casts]:** If  $\models \mathcal{S} \parallel T_1 \Rightarrow T_2$  and  $e \in \llbracket T_1 \rrbracket$  then  $\langle T_1 \xrightarrow{S_1} T_2 \rangle^l e \in \llbracket T_2 \rrbracket$ .

**Proof:** By induction on the sum of the heights of  $T_1$  and  $T_2$ . We go by cases on the shape of the types.

$(T_i = \{x:B \mid e_i\})$  We know that  $e$  goes to blame or a constant. In the former case, the entire term goes to blame by E\_CASTRAISE. Otherwise, we go by E\_CHECKSET and the normalization assumptions in WF\_TYPESET until we run out of types in  $\mathcal{S}$ , at which time we apply E\_CHECKNONE and the normalization assumption in  $\models T_2$ .

$(T_i = T_{i1} \rightarrow T_{i2})$  We know that  $e \in \llbracket T_1 \rightarrow T_2 \rrbracket$ , so it normalizes to some  $e'$ . By cast congruence (Lemma B.14), we know that  $\langle T_1 \xrightarrow{\mathcal{S}} T_2 \rangle^l e$  and  $\langle T_1 \xrightarrow{\mathcal{S}} T_2 \rangle^l e'$  terminate together. We go by cases on the shape of the result  $e' \in \llbracket T_1 \rrbracket$ .

$(e' = \uparrow l')$  We are done by E\_CASTRAISE and Lemma B.11.

$(e' = \lambda x:T_{11}. e_1)$  We have a value. We must show that  $\langle T_{11} \rightarrow T_{12} \xrightarrow{\mathcal{S}} T_{21} \rightarrow T_{22} \rangle^l \lambda x:T_{11}. e_1 \in \llbracket T_{21} \rightarrow T_{22} \rrbracket$ . Let  $e_2 \in \llbracket T_{21} \rrbracket$  be a heedful  $\lambda_H$  result. We can step by E\_UNWRAP and then apply the IH on the smaller domain and codomain types.

$(e' = \langle T_{31} \rightarrow T_{32} \xrightarrow{\mathcal{S}'} T_{11} \rightarrow T_{12} \rangle^{l'} \lambda x:T_{31}. e_1)$  We step by E\_CASTMERGE to:

$$\langle T_{31} \rightarrow T_{32} \xrightarrow{\mathcal{S}' \cup \mathcal{S} \cup \{T_{11} \rightarrow T_{12}\}} T_{21} \rightarrow T_{22} \rangle^l \lambda x:T_{31}. e_1$$

Let  $e_2 \in \llbracket T_{21} \rrbracket$ . We step by E\_UNWRAP, observing that we can use cast congruence (Lemma B.14) to factor the domain and codomain casts, using the IH to handle  $\mathcal{S}$  and  $T_{21} \rightarrow T_{22}$  and the assumptions about  $e'$  to handle the rest.

□

To be able to use our *semantic* cast congruence lemma, we must show that all well typed heedful  $\lambda_H$  terms are in the relation we define; this proof is standard.

#### B.16 Lemma [Strong normalization of heedful terms]:

- $\Gamma \vdash_H e : T$  implies  $\Gamma \models e : T$ ,
- $\vdash_H T$  implies  $\models T$ , and
- $\vdash_H \mathcal{S} \parallel T_1 \Rightarrow T_2$  implies  $\models \mathcal{S} \parallel T_1 \Rightarrow T_2$ .

**Proof:** By mutual induction on the typing derivations.

#### Term typing $\boxed{\Gamma \vdash_H e : T}$

T\_VAR We know by assumption that  $\sigma(x) \in \llbracket T \rrbracket$ .

T\_CONST Evaluation is by reflexivity; we find  $\text{ty}(k) = B$  by assumption.

T\_ABS Let  $\Gamma \models \sigma$ . We must show that  $\lambda x:T_1. \sigma(e_1) \in \llbracket T_1 \rightarrow T_2 \rrbracket$ . Let  $e_2 \in \llbracket T_1 \rrbracket$ . We must show that applying the abstraction to the result yields related values. If  $e_2$  is blame we are done; if not, we step by E\_BETA, to  $\sigma(e_1)[e_2/x]$ . But  $\Gamma, x:T_1 \models \sigma[e_2/x]$ , so we can apply IH (B.16) on  $e_1$ .

T\_OP By IH (B.16) on each argument, either one of the arguments goes to blame, we are done by E\_OPRAISE, or, all of the arguments normalize. We then reduce by E\_OP on to have  $\llbracket op \rrbracket(k_1, \dots, k_n)$ . We have assumed that the denotations of operations agree with their typings in *all* modes, so then  $\llbracket op \rrbracket(k_1, \dots, k_n)$  produces a constant of appropriate base type (and, in fact, refinement) for  $\rightarrow_H$  in particular, and we are done.

T\_APP Let  $\Gamma \models \sigma$ . We must show that  $\sigma(e_1) \sigma(e_2) \in \llbracket T_2 \rrbracket$ . But by IH (B.16) on  $e_1$  and  $e_2$ , we are done directly.

T\_CAST By Lemma B.15, using IH (B.16) on the type set and IH (B.16) on the term.

T\_BLAKE By Lemma B.11.

T\_CHECK By IH (B.16), we know that the active check reduces to a boolean or blame, which then reduces to blame or the appropriate constant  $k$ .

#### Type well formedness $\boxed{\vdash_H T}$

WF\_BASE We can immediately see  $\text{true}[k/x] \in \llbracket \{x:\text{Bool} \mid \text{true}\} \rrbracket$  by reflexivity and definition of constants.

WF\_REFINE By inversion, we know that  $x:\{x:B \mid \text{true}\} \vdash_H e : \{x:\text{Bool} \mid \text{true}\}$ ; by IH (B.16), we find that  $\sigma(e) \in \llbracket \{x:\text{Bool} \mid \text{true}\} \rrbracket$ , i.e., that  $e[e_2/x] \in \llbracket \{x:\text{Bool} \mid \text{true}\} \rrbracket$  for all  $e_2 \in \llbracket \{x:B \mid \text{true}\} \rrbracket$ —which is what we needed to know.

WF\_FUN By IH (B.16) on each of the types.

**Type set well formedness**  $\boxed{\vdash_H S \parallel T_1 \Rightarrow T_2}$

WF\_TypeSet By IH (B.16) on each of the types. □

We define the logical relation in Figure 12. The main difference is that this relation is *symmetric*: classic and heedful  $\lambda_H$  yield blame or values iff the other one does, though the blame labels may be different. The formulations are otherwise the same, and the proof proceeds similarly—though heedful  $\lambda_H$ 's more complicated cast merging leads to some more intricate stepping in the cast lemma.

**B.17 Lemma [Value relation relates only values]:** If  $e_1 \sim_H e_2 : T$  then  $\text{val}_C e_1$  and  $\text{val}_H e_2$ .

**Proof:** By induction on  $T$ . We have  $e_1 = e_2 = k$  when  $T = \{x:B \mid e\}$  (and so we are done by V\_CONST). When  $T = T_1 \rightarrow T_2$ , we have the value derivations as assumptions. □

**B.18 Lemma [Relation implies similarity]:** If  $T_1 \sim_H T_2$  then  $\vdash T_1 \parallel T_2$ .

**Proof:** By induction on  $T_1$ , using S\_REFINE and S\_FUN. □

**B.19 Lemma [Relating classic and heedful casts]:** If  $T_{11} \sim_H T_{21}$  and  $T_{12} \sim_H T_{22}$  and  $\vdash T_{11} \parallel T_{12}$ , then forall  $e_1 \sim_H e_2 : T_{21}$ , we have  $\langle T_{11} \dot{\Rightarrow} T_{12} \rangle^l e_1 \simeq_H \langle T_{21} \dot{\Rightarrow} T_{22} \rangle^{l'} e_2 : T_{22}$ .

**Proof:** By induction on the sum of the heights of  $T_{21}$  and  $T_{22}$ . By Lemma B.18, we know that  $\vdash T_{11} \parallel T_{21}$  and  $\vdash T_{12} \parallel T_{22}$ ; by Lemma A.7, we know that  $\vdash T_{21} \parallel T_{22}$ . We go by cases on  $T_{22}$ . The heedful term first steps by E\_TypeSet, replacing its  $\bullet$  annotation with an empty set.

( $T_{22} = \{x:B \mid e_{22}\}$ ) It must be the case (by similarity) that all of the other types are also refinements. Moreover, it must be that case that  $e_1 = e_2 = k$ .

Classic steps by E\_CHECKNONE, while heedful steps by E\_CHECKEMPTY. Since  $e_1 \sim_H e_2 : T_{21} = \{x:B \mid e_{21}\}$ , we can find that  $e_1 \sim_H e_2 : \{x:B \mid \text{true}\}$  trivially. Then, since  $\{x:B \mid e_{12}\} \sim_H \{x:B \mid e_{22}\}$ , we know that  $e_{12}[k/x] \simeq_H e_{22}[k/x] : \{x:\text{Bool} \mid \text{true}\}$ .

If  $e_{12}[k/x] \rightarrow_C^* \uparrow l'$ , then  $e_{22}[k/x] \rightarrow_H^* \uparrow l''$ , and both terms reduce to blame by E\_CHECKINNER and E\_CHECKRAISE—this completes the proof. If not, then both predicates reduce to a boolean together. If they reduce to **false**, then both terms eventually reduces to  $\uparrow l$  via E\_CHECKINNER and E\_CHECKFAIL, and we are done. If they both go to **true**, then both sides step by E\_CHECKINNER and E\_CHECKOK to yield  $k$ , and we can find  $k \sim_H k : \{x:B \mid e_{22}\}$  easily—we have a derivation for  $e_{22}[k/x] \rightarrow_H^* \text{true}$  handy.

( $T_{22} = T_{221} \rightarrow T_{222}$ ) By Lemma B.17, we know that  $\text{val}_C e_1$  and  $\text{val}_H e_2$ . So the classic side is a value  $e_{11}$  (by V\_PROXYC), while the heedful side either steps by E\_CASTMERGE to produce a function proxy  $e_{21}$ , or immediately has one, depending on the shape of  $e_2$ : an abstraction immediately yields a value by V\_PROXYH, or E\_CASTMERGE for a function proxy (again yielding a value by V\_PROXYF).

We must now show that  $e_{11} \sim_H e_{21} : T_{221} \rightarrow T_{222}$ , knowing that  $e_1 \sim_H e_2 : T_{211} \rightarrow T_{212}$ . Let  $e_{12} \sim_H e_{22} : T_{221}$  be given. On the classic side, we step by E\_UNWRAP to find  $\langle T_{112} \dot{\Rightarrow} T_{122} \rangle^l (e_1 (\langle T_{121} \dot{\Rightarrow} T_{111} \rangle^l e_{12}))$ . (Recall that the annotations are all empty.)

We now go by cases on whether or not  $e_2$  had to take a step to become a value:

(V\_PROXYH) We have  $e_{21} = \langle T_{211} \rightarrow T_{212} \xrightarrow{\emptyset} T_{221} \rightarrow T_{222} \rangle^l \lambda x:T_{211}. e'_2$  since  $e_2 = \lambda x:T_{211}. e'_2$ . We must show that:

$$\langle T_{112} \dot{\Rightarrow} T_{122} \rangle^l (e_1 (\langle T_{121} \dot{\Rightarrow} T_{111} \rangle^l e_{12})) \simeq_H (\langle T_{211} \rightarrow T_{212} \xrightarrow{\emptyset} T_{221} \rightarrow T_{222} \rangle^l \lambda x:T_{211}. e'_2) e_{22} : T_{222}$$

The heedful side steps by E\_UNWRAP with an empty type set, yielding  $\langle T_{212} \dot{\Rightarrow} T_{222} \rangle^l ((\lambda x:T_{211}. e'_2) (\langle T_{221} \dot{\Rightarrow} T_{211} \rangle^l e_{22}))$ .

By the IH, we know that  $\langle T_{121} \dot{\Rightarrow} T_{111} \rangle^l e_{12} \simeq_H \langle T_{221} \dot{\Rightarrow} T_{211} \rangle^l e_{22} : T_{211}$ . If we get blame on both sides, we are done immediately by the appropriate E...RAISE rules. If not, we get values  $e'_{12} \sim_H e'_{22} : T_{211}$ . We know by assumption that  $e_1 e'_{12} \simeq_H e_2 e'_{22} : T_{212}$ ; again, blame on finishes this case. So suppose both sides go to values  $e''_{12} \sim_H e''_{22} : T_{212}$ . By the IH, we know that  $\langle T_{112} \dot{\Rightarrow} T_{122} \rangle^l e'_{12} \simeq_H \langle T_{212} \dot{\Rightarrow} T_{222} \rangle^l e''_{22} : T_{222}$ , and we are done.

(E\_CASTMERGE) We have  $e_{21} = \langle T_{31} \rightarrow T_{32} \xrightarrow{\mathcal{S} \cup \{T_{211} \rightarrow T_{212}\}} T_{221} \rightarrow T_{222} \rangle^l \lambda x:T_{31}. e'_2$  since  $e_2 = \langle T_{31} \rightarrow T_{32} \xrightarrow{\mathcal{S}} T_{211} \rightarrow T_{212} \rangle^{l'} \lambda x:T_{31}. e'_2$ . We must show that:

$$\langle T_{112} \dot{\Rightarrow} T_{122} \rangle^l (e_1 (\langle T_{121} \dot{\Rightarrow} T_{111} \rangle^l e_{12})) \simeq_F (\langle T_{31} \rightarrow T_{32} \xrightarrow{\mathcal{S} \cup \{T_{211} \rightarrow T_{212}\}} T_{221} \rightarrow T_{222} \rangle^l \lambda x:T_{31}. e'_2) e_{22} : T_{222}$$



The right hand steps by E\_UNWRAP, yielding  $\langle T_{32} \xrightarrow{\text{cod}(\mathcal{S}) \cup \{T_{212}\}} T_{222} \rangle^l ((\lambda x:T_{31}. e'_2) (\langle T_{221} \xrightarrow{\text{dom}(\mathcal{S}) \cup \{T_{211}\}} T_{31} \rangle^l e_{22}))$ . We must show that this heedful term is related to the classic term  $\langle T_{112} \xrightarrow{\bullet} T_{122} \rangle^l (e_1 (\langle T_{121} \xrightarrow{\bullet} T_{111} \rangle^l e_{12}))$ .

We must now make a brief digression to examine the behavior of the cast that was eliminated by E\_CASTMERGE. We know by the IH that  $\langle T_{121} \xrightarrow{\bullet} T_{111} \rangle^l e_{12} \simeq_{\text{H}} \langle T_{221} \xrightarrow{\bullet} T_{211} \rangle^l e_{22} : T_{211}$ , so both sides go to blame or to values  $e'_{12} \sim_{\text{H}} e'_{22} : T_{211}$ . By Lemma B.14 with  $\text{dom}(\mathcal{S})$  as the type set, we can find that  $\langle T_{211} \xrightarrow{\bullet} T_{31} \rangle^l e'_{22} \rightarrow_{\text{H}}^* e''_{22}$  implies  $\langle T_{211} \xrightarrow{\text{dom}(\mathcal{S})} T_{31} \rangle^l (\langle T_{221} \xrightarrow{\bullet} T_{211} \rangle^l e_{22}) \rightarrow_{\text{H}}^* e''_{22}$ . But then we have that  $\langle T_{211} \xrightarrow{\text{S}} T_{31} \rangle^l (\langle T_{221} \xrightarrow{\bullet} T_{211} \rangle^l e_{22}) \rightarrow_{\text{H}} \langle T_{211} \xrightarrow{\text{dom}(\mathcal{S}) \cup \{T_{221}\}} T_{31} \rangle^l e_{22}$ , so we then know that  $\langle T_{221} \xrightarrow{\text{dom}(\mathcal{S}) \cup \{T_{221}\}} T_{31} \rangle^l e_{22} \rightarrow_{\text{H}}^* e''_{22}$ , just as if it were applied to  $e'_{22}$ .

Now we can return to the meat of our proof. If  $\langle T_{121} \xrightarrow{\bullet} T_{111} \rangle^l e_{12} \rightarrow_{\text{H}}^* \uparrow l'$ , we are done—so must the heedful side (albeit possibly at a different blame label). If it reduces to a value  $e'_{12}$ , then we are left considering the term  $\langle T_{112} \xrightarrow{\bullet} T_{122} \rangle^l (e_1 e'_{12})$  on the classic side. We know that  $e_1 \sim_{\text{H}} e_2 : T_{21}$ . Unfolding the definition of  $e_2$ , this means that  $e_1 e'_{12} \simeq_{\text{F}} \langle T_{32} \xrightarrow{\bullet} T_{212} \rangle^l ((\lambda x:T_{31}. e'_2) (\langle T_{211} \xrightarrow{\bullet} T_{31} \rangle^l e'_{22})) : T_{212}$ . If the classic side produces blame, so must the heedful side and we are done, as indicated in the digression above. If not, then both sides produce values. For these terms to produce values, it must be the case that (a) the domain cast on the heedful side produces a value, (b) the heedful function produces a value given that input, and (c) the heedful codomain cast produces a value. Now, we know from our digression above that  $\langle T_{211} \xrightarrow{\text{dom}(\mathcal{S})} T_{31} \rangle^l e'_{22}$  and  $\langle T_{211} \xrightarrow{\text{dom}(\mathcal{S})} T_{31} \rangle^l e_{22}$  reduce to the exact same value,  $e''_{22}$ . So if  $\langle T_{32} \xrightarrow{\text{cod}(\mathcal{S})} T_{212} \rangle^l ((\lambda x:T_{31}. e'_2) (\langle T_{211} \xrightarrow{\text{dom}(\mathcal{S})} T_{31} \rangle^l e'_{22})) \rightarrow_{\text{H}}^* e''_{22}$  then we can also see

$$\langle T_{32} \xrightarrow{\text{cod}(\mathcal{S})} T_{212} \rangle^l ((\lambda x:T_{31}. e'_2) (\langle T_{211} \xrightarrow{\text{dom}(\mathcal{S})} T_{31} \rangle^l e_{22})) \rightarrow_{\text{H}}^* \langle T_{32} \xrightarrow{\bullet} T_{212} \rangle^l e_{32} \rightarrow_{\text{H}}^* e''_{22}.$$

We have shown that the domains and then the applied inner functions are equivalent. It now remains to show that

$$\langle T_{112} \xrightarrow{\bullet} T_{122} \rangle^l e'_{11} \simeq_{\text{H}} \langle T_{32} \xrightarrow{\text{cod}(\mathcal{S}) \cup \{T_{212}\}} T_{222} \rangle^l ((\lambda x:T_{31}. e'_2) (\langle T_{221} \xrightarrow{\bullet} T_{31} \rangle^l e_{22})) : T_{222}$$

We write the *entire* heedful term to highlight the fact that we cannot freely apply congruence, but must instead carefully apply cast congruence (Lemma B.14) as we go.

By the IH, we know that either  $\langle T_{112} \xrightarrow{\bullet} T_{122} \rangle^l e'_{11}$  and  $\langle T_{212} \xrightarrow{\bullet} T_{222} \rangle^l e'_{22}$  go to blame (perhaps with different labels) or to values. In the former case we are done; in the latter case, we already know that  $\langle T_{32} \xrightarrow{\text{cod}(\mathcal{S})} T_{212} \rangle^l e_{32} \rightarrow_{\text{H}}^* e''_{22}$ , so we can apply cast congruence (Lemma B.14) with the empty type set to see that if  $\langle T_{212} \xrightarrow{\bullet} T_{222} \rangle^l e'_{22} \rightarrow_{\text{H}}^* e''_{22}$  then  $\langle T_{212} \xrightarrow{\bullet} T_{222} \rangle^l (\langle T_{32} \xrightarrow{\text{cod}(\mathcal{S})} T_{212} \rangle^l e_{32}) \rightarrow_{\text{H}}^* e''_{22}$ . But we know that  $\langle T_{212} \xrightarrow{\bullet} T_{222} \rangle^l (\langle T_{32} \xrightarrow{\text{cod}(\mathcal{S})} T_{212} \rangle^l e_{32}) \rightarrow_{\text{H}} \langle T_{32} \xrightarrow{\text{cod}(\mathcal{S}) \cup \{T_{212}\}} T_{222} \rangle^l e_{32}$  deterministically, so we then we know that  $\langle T_{32} \xrightarrow{\text{cod}(\mathcal{S}) \cup \{T_{212}\}} T_{222} \rangle^l e_{32} \rightarrow_{\text{H}}^* e''_{22}$ . Since  $\langle T_{32} \xrightarrow{\text{cod}(\mathcal{S}) \cup \{T_{212}\}} T_{222} \rangle^l ((\lambda x:T_{31}. e'_2) (\langle T_{221} \xrightarrow{\bullet} T_{31} \rangle^l e_{22})) \rightarrow_{\text{H}}^* \langle T_{32} \xrightarrow{\bullet} T_{222} \rangle^l e_{32}$ , we have shown that the classic term and heedful term reduce to values  $e'_{12} \sim_{\text{H}} e''_{22} : T_{222}$ , and we are done.  $\square$

## B.20 Lemma [Relating classic and heedful source programs]:

1. If  $\Gamma \vdash_{\text{C}} e : T$  as a source program then  $\Gamma \vdash e \simeq_{\text{H}} e : T$ .
2. If  $\vdash_{\text{C}} T$  as a source program then  $T \sim_{\text{H}} T$ .

**Proof:** By mutual induction on the typing derivations.

### Term typing $\boxed{\Gamma \vdash_{\text{C}} e : T}$

**T\_VAR** We know by assumption that  $\delta_1(x) \sim_{\text{H}} \delta_2(x) : T$ .

**T\_CONST** Since we are dealing with a source program,  $T = \{x:B \mid \text{true}\}$ . We have immediately that  $\text{ty}(k) = B$  and  $\text{true}[k/x] \simeq_{\text{H}} \text{true}[k/x] : \{x:\text{Bool} \mid \text{true}\}$ , so  $k \simeq_{\text{H}} k : \{x:B \mid \text{true}\}$ .

**T\_ABS** Let  $\Gamma \models_{\text{H}} \delta$ . We must show that  $\lambda x:T_1. \delta_1(e_1) \sim_{\text{H}} \lambda x:T_1. \delta_2(e_1) : T_1 \rightarrow T_2$ . Let  $e_2 \sim_{\text{H}} e'_2 : T_1$ . We must show that applying the abstractions to these values yields related values. Both sides step by E\_BETA, to  $\delta_1(e_1)[e_2/x]$  and  $\delta_2(e_1)[e'_2/x]$ , respectively. But  $\Gamma, x:T_1 \models_{\text{H}} \delta[e_2, e'_2/x]$ , so we can apply IH (1) on  $e_1$  showing the two sides reduce to related results.

**T\_OP** By IH (1) on each argument, either one of the arguments goes to blame (in both calculi), and we are done by E\_OPRAISE, or all of the arguments reduce to related values. Since  $\text{ty}(op)$  is first order, these values must be related at refined base types, which means that they are in fact all equal constants. We then reduce by E\_OP on both sides to have  $\llbracket op \rrbracket(k_1, \dots, k_n)$ . We have assumed that the denotations of operations agree with their typings in *all* modes, so then  $\llbracket op \rrbracket(k_1, \dots, k_n)$  satisfies the refinement for  $\rightarrow_H$  in particular, and we are done.

**T\_APP** Let  $\Gamma \models_H \delta$ . We must show that  $\delta_1(e_1) \delta_1(e_2) \simeq_H \delta_2(e_1) \delta_2(e_2) : T_2$ . But by IH (1) on  $e_1$  and  $e_2$ , we are done directly.

**T\_CAST** We know that the annotation is  $\bullet$ , since we are dealing with a source term. Let  $\Gamma \models_H \delta$ . By IH (1) on  $e'$ , we know that  $\delta_1(e') \simeq_H \delta_2(e') : T_1$ , either  $\delta_1(e') \rightarrow_C^* \uparrow l'$  and  $\delta_2(e') \rightarrow_H^* \uparrow l''$  (and we are done) or  $\delta_1(e')$  and  $\delta_2(e')$  reduce to values  $e_1 \sim_H e_2 : T_1$ . By Lemma B.19 (using IH (2) on the types), we know that  $\langle T_1 \dot{\Rightarrow} T_2 \rangle^l e_1 \sim_H \langle T_1 \dot{\Rightarrow} T_2 \rangle^l e_2 : T_2$ , so each side must reduce to a result  $e'_1 \sim_H e'_2 : T_2$ . We have cast congruence on the classic side straightforwardly, finding:

$$\langle T_1 \dot{\Rightarrow} T_2 \rangle^l \delta_1(e') \rightarrow_C^* \langle T_1 \dot{\Rightarrow} T_2 \rangle^l e_1 \rightarrow_C^* e'_1$$

On the heedful side, we can apply our derived cast congruence (Lemma B.14) to find that  $\langle T_1 \dot{\Rightarrow} T_2 \rangle^l e_2 \rightarrow_H^* e'_2$  and  $\delta_2(e') \rightarrow_H^* e_2$  imply that  $\langle T_1 \dot{\Rightarrow} T_2 \rangle^l \delta_2(e') \rightarrow_H^* e'_2$ .

**T\_BLAKE** Contradiction—doesn't appear in source programs. Though in fact it is in the relation, since  $\uparrow l \simeq_H \uparrow l' : T$  for any  $l, l'$ , and  $T$ .

**T\_CHECK** Contradiction—doesn't appear in source programs.

### Type well formedness $\boxed{\vdash_C T}$

**WF\_BASE** We can immediately see  $\text{true}[k/x] \simeq_H \text{true}[k/x] : \{x:\text{Bool} \mid \text{true}\}$  for any  $k \sim_H k : \{x:B \mid \text{true}\}$ , i.e., any  $k$  such that  $\text{ty}(k) = B$ .

**WF\_REFINE** By inversion, we know that  $x:\{x:B \mid \text{true}\} \vdash_C e : \{x:\text{Bool} \mid \text{true}\}$ ; by IH (1), we find that  $\delta_1(e) \simeq_H \delta_2(e) : \{x:\text{Bool} \mid \text{true}\}$ , i.e., that  $e[e_1/x] \simeq_H e[e_2/x] : \{x:\text{Bool} \mid \text{true}\}$  for all  $e_1 \sim_H e_2 : \{x:B \mid \text{true}\}$ —which is what we needed to know.

**WF\_FUN** By IH (2) on each of the types.

□

We have investigated two alternatives to the formulation here: type set optimization and invariants that clarify the role of type sets.

First, we can imagine a system that optimizes the type set of  $\langle T_1 \dot{\Rightarrow} T_2 \rangle^l$  such that  $T_1$  and  $T_2$  don't appear in  $S$ —taking advantage of idempotence not only for the source type (Lemma B.9) but also for the target type. This change complicates the theory but doesn't give any stronger theorems. Nevertheless, such an optimization would be a sensible addition to an implementation.

Second, our proof relates source programs, which start with empty annotations. In fact, all of the reasoning about type sets is encapsulated in our proof cast congruence (Lemma B.14). We could define a function from heedful  $\lambda_H$  to classic  $\lambda_H$  that unrolls type sets according to the **choose** function. While this proof would offer a direct understanding of heedful  $\lambda_H$  type sets in terms of the classic semantics, it wouldn't give us a strong property—it degenerates to our proof in the empty type set case.

## B.3 Relating classic and eidetic manifest contracts

**B.21 Lemma [Idempotence of coercions]:** If  $\emptyset \vdash_E k : \{x:B \mid e_1\}$  and  $\vdash_E r_1 \triangleright r_2 \parallel \{x:B \mid e_1\} \Rightarrow \{x:B \mid e_2\}$ , then for all  $\text{result}_E e$ , it is the case that  $\langle \{x:B \mid e_1\}^{r_1 \triangleright (r_2 \parallel \{x:B \mid e_1\})} \{x:B \mid e_2\} \rangle^\bullet k \rightarrow_E^* e$  iff  $\langle \{x:B \mid e_1\}^{r_1 \triangleright r_2} \{x:B \mid e_2\} \rangle^\bullet k \rightarrow_E^* e$ .

**Proof:** By induction on their evaluation derivations: the only difference is that the latter derivation performs some extra checks that are implied by  $e_1[k/x] \rightarrow_E^* \text{true}$ —which we already know to hold. □

As before, cast congruence is the key lemma in our proof—in this case, the strongest property we have: reduction to identical results.

**B.22 Lemma [Cast congruence (single step)]:** If

$$- \emptyset \vdash_E e_1 : T_1 \text{ and } \vdash_E c \parallel T_1 \Rightarrow T_2 \text{ (and so } \emptyset \vdash_E \langle T_1 \dot{\Rightarrow} T_2 \rangle^\bullet e_1 : T_2),$$

–  $e_1 \longrightarrow_E e_2$  (and so  $\emptyset \vdash_E e_2 : T_1$ ),

then for all  $\text{result}_E e$ , we have  $\langle T_1 \xRightarrow{c} T_2 \rangle^\bullet e_1 \longrightarrow_E^* e$  iff  $\langle T_1 \xRightarrow{c} T_2 \rangle^\bullet e_2 \longrightarrow_E^* e$ .

**Proof:** By cases on the step taken to find  $e_1 \longrightarrow_E e_2$ .

There are two groups of reductions: straightforward merge-free reductions and merging reductions. In many cases, we simply show confluence, which implies the coterminality at identical values in our deterministic semantics (Lemma A.28).

**Merge-free reductions** In these cases, we apply  $E\_COERCEINNER$  and whatever rule derived  $e_1 \longrightarrow_E e_2$  to find that  $\langle T_1 \xRightarrow{c} T_2 \rangle^\bullet e_1 \longrightarrow_E \langle T_1 \xRightarrow{c} T_2 \rangle^\bullet e_2$ , i.e.,  $e = \langle T_1 \xRightarrow{c} T_2 \rangle^\bullet e_2$ .

- (E\_BETA) By  $E\_COERCEINNER$  and  $E\_BETA$ .
- (E\_OP) By  $E\_COERCEINNER$  and  $E\_OP$ .
- (E\_UNWRAP) By  $E\_COERCEINNER$  and  $E\_UNWRAP$ .
- (E\_APPL) By  $E\_COERCEINNER$  with  $E\_APPL$ .
- (E\_APPR) By  $E\_COERCEINNER$  with  $E\_APPR$ .
- (E\_APPRAISEL) By  $E\_COERCEINNER$  with  $E\_APPRAISEL$ ; then by  $E\_CASTRAISE$  on both sides.
- (E\_APPRAISER) By  $E\_COERCEINNER$  with  $E\_APPRAISER$ ; then by  $E\_CASTRAISE$  on both sides.
- (E\_COERCE) By  $E\_COERCEINNER$  and  $E\_COERCE$ .
- (E\_STACKDONE) By  $E\_COERCEINNER$  and  $E\_STACKDONE$ .
- (E\_STACKPOP) By  $E\_COERCEINNER$  and  $E\_STACKRAISE$ .
- (E\_STACKINNER) By  $E\_COERCEINNER$  and  $E\_STACKRAISE$ .
- (E\_STACKRAISE) By  $E\_COERCEINNER$  and  $E\_STACKRAISE$ .
- (E\_CHECKOK) By  $E\_COERCEINNER$  and  $E\_CHECKOK$ .
- (E\_CHECKFAIL) By  $E\_COERCEINNER$  and  $E\_CHECKRAISE$ .
- (E\_CHECKFAIL) By  $E\_COERCEINNER$  and  $E\_CHECKRAISE$ .
- (E\_OPINNER) By  $E\_COERCEINNER$  and  $E\_OPINNER$ .
- (E\_OPRAISE) By  $E\_COERCEINNER$  and  $E\_OPRAISE$ .

**Merging reductions** In these cases, some coercion in  $e_1$  reduces when we step  $e_1 \longrightarrow_E e_2$ , but merges when we consider  $\langle T_1 \xRightarrow{c} T_2 \rangle^\bullet e_1$ . We must show that the merged term and  $\langle T_1 \xRightarrow{c} T_2 \rangle^\bullet e_2$  eventually meet at some common term  $e$ . It's convenient to renumber the types and coercions, so we consider  $\langle T_2 \xRightarrow{c_2} T_3 \rangle^\bullet e_1$  where  $e_1 = \langle T_1 \xRightarrow{c_1} T_2 \rangle^\bullet e \longrightarrow_E e_2$  for some  $e$ .

(E\_COERCESTACK) We have  $e_1 = \langle T_1 \xRightarrow{r_1} T_2 \rangle^\bullet k$  and  $e_2 = \langle T_2, ?, r_1, k, k \rangle^\bullet$ . Since our term is well typed, we know that  $c_2$  is some  $r_2$ . In the original, unreduced term with  $e_1$ , we step to  $\langle T_1 \xRightarrow{r_1 \triangleright r_2} T_3 \rangle^\bullet k$  by  $E\_CASTMERGE$ ; we then step by  $E\_COERCESTACK$  to  $\langle T_3, ?, r_1 \triangleright r_2, k, k \rangle^\bullet$ . Call this term  $e'_1$ .

In the reduced term with  $e_2$ , we have  $\langle T_2 \xRightarrow{r_2} T_3 \rangle^\bullet \langle T_2, ?, r_1, k, k \rangle^\bullet$ . Call this term  $e'_2$ .

We must show that  $e'_1$  reduces to a given result iff  $e'_2$  does.

The term  $e'_1$  evaluates by running through the checks in  $r_1 \triangleright r_2$ , raising blame if a check fails or returning  $k$  if they all succeed.

The term  $e'_2$  evaluates by running through the checks in  $r_1$ , raising blame if a check fails or returning  $k$  if they all succeed, eventually reducing to  $\langle T_3, ?, r_2, k, k \rangle^\bullet$  in that case. This term, similarly, reduces to  $k$  or  $\uparrow l$  if a given check fails. Note that types are preserved by  $\triangleright$  from left-to-right, so if there is a type that fails in  $r_1$ , it fails in  $r_1 \triangleright r_2$ , as well. So  $e'_2$  fails in the first set of checks iff  $e'_1$  fails in the first half of  $r_1 \triangleright r_2$ .

Now we must consider those checks in  $r_2$ . It may be that there are types in  $r_2$  that were subsumed by a check in  $r_1$ : these types are re-checked in  $e'_2$  but not in  $e'_1$ , since the latter merges coercions while the former doesn't. We can show that these second checks are redundant by idempotence (Lemma B.21), allowing us to conclude that  $e'_1$  and  $e'_2$  also behave the same on the second half of the checks, and therefore the two terms both go to the same blame label or to the same constant.

(E\_COERCEINNER)  $e_1 = \langle T_1 \xRightarrow{c_1} T_2 \rangle^\bullet e'_1$  and  $e_2 = \langle T_1 \xRightarrow{c_1} T_2 \rangle^\bullet e''_1$  such that  $e_1 \longrightarrow_E e''_1$  and  $e_1$  isn't a coercion term. Both sides reduce to the common term  $\langle T_1 \xRightarrow{c_1 \triangleright c_2} T_3 \rangle^\bullet e'_1$ . We step  $\langle T_2 \xRightarrow{c_2} T_3 \rangle^\bullet (\langle T_1 \xRightarrow{c_1} T_2 \rangle^\bullet e'_1)$  first by  $E\_CASTMERGE$  to  $\langle T_1 \xRightarrow{c_1 \triangleright c_2} T_3 \rangle^\bullet e'$ , after which we can apply  $E\_COERCEINNER$ . We step  $\langle T_2 \xRightarrow{c_2} T_3 \rangle^\bullet (\langle T_1 \xRightarrow{c_1} T_2 \rangle^\bullet e'_1)$  directly to  $e$  by  $E\_CASTMERGE$ .

(E\_CASTMERGE)  $e_1 = \langle T_1 \xRightarrow{c_1} T_2 \rangle^\bullet (\langle T_0 \xRightarrow{c_0} T_1 \rangle^\bullet e')$  and  $e_2 = \langle T_0 \xRightarrow{c_0 \triangleright c_1} T_2 \rangle^\bullet e'$ . The common term here is  $\langle T_0 \xRightarrow{c_0 \triangleright c_1 \triangleright c_2} T_3 \rangle^\bullet e'$ : we step the left-hand side by  $E\_CASTMERGE$  twice, and  $e_2$  only once.

(E\_COERCERAISE) We have  $e_1 = \langle T_1 \xrightarrow{c_1} T_2 \rangle^\bullet \uparrow l$  and  $e_2 = \uparrow l$ . Both reduce to the common term  $\uparrow l$ . The former first steps by E\_CASTMERGE, and then both step by E\_COERCERAISE.

□

**B.23 Lemma [Cast congruence]:** If

- $\emptyset \vdash_E e_1 : T_1$  and  $\vdash_E c \parallel T_1 \Rightarrow T_2$  (and so  $\emptyset \vdash_E \langle T_1 \xrightarrow{c} T_2 \rangle^\bullet e_1 : T_2$ ),
- $e_1 \xrightarrow{*}_E e_2$  (and so  $\emptyset \vdash_E e_2 : T_1$ ),

then there exists an  $e$  such that  $\langle T_1 \xrightarrow{c} T_2 \rangle^\bullet e_1 \xrightarrow{*}_E e$  and  $\langle T_1 \xrightarrow{c} T_2 \rangle^\bullet e_2 \xrightarrow{*}_E e$ . Diagrammatically:

$$\begin{array}{ccc}
 e_1 & \xrightarrow{\quad} & e_2 \\
 \Downarrow & & \\
 \langle T_1 \xrightarrow{\bullet} T_2 \rangle^l e_1 & & \langle T_1 \xrightarrow{\bullet} T_2 \rangle^l e_2 \\
 \searrow & & \swarrow \\
 \text{result}_E e & & \text{result}_E e
 \end{array}$$

**Proof:** By induction on the derivation  $e \xrightarrow{*}_E e_1$ , using the single-step cast congruence (Lemma B.22). □

Our proof strategy is as follows: we show that the casts between related types are applicative, and then we show that well typed source programs in classic  $\lambda_H$  are logically related to their translation. Our definitions are in Figure 12. Our logical relation is *blame-exact*. Like our proofs relating forgetful and heedful  $\lambda_H$  to classic  $\lambda_H$ , we use the space-efficient semantics in the refinement case and use space-efficient type indices.

**B.24 Lemma [Similar casts are logically related]:** If  $T_1 \sim_E T'_1$  and  $T_2 \sim_E T'_2$  and  $e_1 \sim_E e_2 : T_1$ , then  $\langle T_1 \xrightarrow{\bullet} T_2 \rangle^l e_1 \simeq_E \langle T'_1 \xrightarrow{\bullet} T'_2 \rangle^l e_2 : T_2$ .

**Proof:** By induction on the invariant relation, using coercion congruence in the function case when  $e_2$  is a function proxy. We always step first by E\_COERCE on the right to  $\langle T_1^{\text{coerce}(T'_1, T'_2, l)} \xrightarrow{\bullet} T'_2 \rangle^\bullet e_2$ .

(A\_REFINE) Let  $e'_1 \sim_E e'_2 : \{x:B \mid e'_1\}$ , we know that  $e'_1 = e'_2 = k$  such that  $e'_1[k/x] \xrightarrow{*}_E \text{true}$ . In classic  $\lambda_H$ , we step by E\_CHECKNONE to  $\langle \{x:B \mid e_2\}, e_2[k/x], k \rangle^l$ ; in eidetic  $\lambda_H$ , we step by E\_COERCE and then E\_COERCESTACK to  $\langle \{x:B \mid e'_2\}, \cdot, \{x:B \mid e'_2\}^l, k, k \rangle^\bullet$ , and then by E\_STACKPOP to

$$\langle \{x:B \mid e'_2\}, \cdot, \text{nil}, k, \langle \{x:B \mid e'_2\}, e'_2[k/x], k \rangle^l \rangle^\bullet$$

Since  $k \sim_E k : \{x:B \mid \text{true}\}$  by definition and reflexivity of  $\xrightarrow{*}_E$ , we know that  $e_2[k/x] \simeq_E e'_2[k/x] : \{x:\text{Bool} \mid \text{true}\}$ . If the predicates step to a blame label (the same one!), then both terms raise that label (by E\_CHECKRAISE, with an added E\_STACKRAISE on the right). Similarly, if the predicates go to *false*, then both sides raise  $\uparrow l$  by E\_CHECKFAIL (followed by the same steps as for inner blame). Finally, if the predicates both go to *true*, then both checks return  $k$ . After stepping by E\_STACKDONE on the right, we find that both terms reduce to  $k$  and that  $e'_2[k/x] \xrightarrow{*}_E \text{true}$ .

(A\_FUN) We have  $T_{11} \rightarrow T_{12} \sim_E T'_{11} \rightarrow T'_{12}$  and  $T_{21} \rightarrow T_{22} \sim_E T'_{21} \rightarrow T'_{22}$ . Let  $e_1 \sim_E e_2 : T'_{11} \rightarrow T'_{12}$ . The classic side is a value, by V\_PROXYC. The eidetic  $\lambda_H$  term is:

$$\langle T'_{11} \rightarrow T'_{12} \xrightarrow{c_1 \mapsto c_2} T'_{21} \rightarrow T'_{22} \rangle^\bullet e_2$$

How this term steps depends on the shape of the value  $e_2$ : either  $e_2$  is an abstraction  $\lambda x:T. e$  and we have a value by V\_PROXYE, or it is a function proxy  $\langle T_{01} \rightarrow T_{02} \xrightarrow{c'_1 \mapsto c'_2} T'_{11} \rightarrow T'_{12} \rangle^\bullet \lambda x:T_{01}. e$  and we step by E\_CASTMERGE.

(V\_PROXYE) We step to  $\langle T'_{11} \rightarrow T'_{12} \xrightarrow{c_1 \mapsto c_2} T'_{21} \rightarrow T'_{22} \rangle^\bullet e_2$ . Let  $e'_1 \sim_E e'_2 : T'_{21}$  be given. Both sides unwrap, giving us  $\langle T_{12} \xrightarrow{\bullet} T_{22} \rangle^l (e_1 (\langle T_{21} \xrightarrow{\bullet} T_{11} \rangle^l e'_1))$  on the classic side and  $\langle T'_{12} \xrightarrow{\bullet} T'_{22} \rangle^\bullet (e_2 (\langle T'_{21} \xrightarrow{\bullet} T'_{12} \rangle^\bullet e'_2))$ . By the IH, the arguments are related and reduce to related results (by expansion via E\_COERCE and the observation that  $\text{coerce}(T'_{21}, T'_{11}, l) = c_1$ ). Blame (at the same label!) aborts the computation. If the arguments produce values, then we apply our assumption that  $e_1 \sim_E e_2 : T'_{11} \rightarrow T'_{12}$ , so  $e_1 e'_1 \simeq_E e_2 e'_2 : T'_{12}$ . Again, blame (at the same label!) aborts early. A value flows to the related codomain casts, and we are done by the IH and E\_COERCE-expansion (observing  $\text{coerce}(T'_{12}, T'_{22}, l) = c_2$ ).

(E\_CASTMERGE) We step to  $\langle T'_{01} \rightarrow T'_{02} \xrightarrow{(c_1 \triangleright c'_1) \mapsto (c'_2 \triangleright c_2)} T'_{21} \rightarrow T'_{22} \rangle^\bullet \lambda x : T'_{01}. e$ . Let  $e'_1 \sim_E e'_2 : T'_{21}$  be given. Both sides unwrap as above. On the classic side, we have the same term as before:  $\langle T_{12} \xrightarrow{\bullet} T_{22} \rangle^l (e_1 (\langle T_{21} \xrightarrow{\bullet} T_{11} \rangle^l e'_1))$ . On the eidetic side, we have some extra coercions:  $\langle T'_{02} \xrightarrow{c'_2 \triangleright c_2} T'_{22} \rangle^\bullet ((\lambda x : T'_{01}. e) (\langle T'_{21} \xrightarrow{c_1 \triangleright c'_1} T'_{01} \rangle^\bullet e'_1))$ . We use coercion congruence to resolve these, and see that both terms behave the same.

Considering the argument, we can factor it out to the term  $\langle T'_{11} \xrightarrow{c'_1} T'_{01} \rangle^\bullet ((\langle T'_{21} \xrightarrow{c_1} T'_{11} \rangle^\bullet e'_2))$ . We know that  $\langle T_{21} \xrightarrow{\bullet} T_{11} \rangle^l e'_1 \simeq_E \langle T'_{21} \xrightarrow{c_1} T'_{11} \rangle^\bullet e'_2 : T'_{11}$  by the IH (with E\_COERCE-expansion, observing that  $\text{coerce}(T'_{21}, T'_{11}, l) = c_1$ ), so they reduce to related results  $e'_1 \simeq_E e'_2 : T'_{11}$ . By coercion congruence (Lemma B.23), we know that  $\langle T'_{11} \xrightarrow{c'_1} T'_{01} \rangle^\bullet e'_2$  and  $\langle T'_{11} \xrightarrow{c'_1} T'_{01} \rangle^\bullet ((\langle T'_{21} \xrightarrow{c_1} T'_{11} \rangle^\bullet e'_2))$  behave identically. We can make a similar observation in the codomain: factoring out to  $\langle T'_{12} \xrightarrow{c_2} T'_{22} \rangle^\bullet ((\langle T'_{02} \xrightarrow{c'_2} T'_{12} \rangle^\bullet ((\lambda x : T'_{01}. e) (\langle T'_{11} \xrightarrow{c'_1} T'_{01} \rangle^\bullet ((\langle T'_{21} \xrightarrow{c_1} T'_{11} \rangle^\bullet e'_2))))))$ , we know that this term is equivalent to  $\langle T'_{12} \xrightarrow{c_2} T'_{22} \rangle^\bullet ((\langle T'_{02} \xrightarrow{c'_2} T'_{12} \rangle^\bullet ((\lambda x : T'_{01}. e) (\langle T'_{11} \xrightarrow{c'_1} T'_{01} \rangle^\bullet e'_2))))$ ; by assumption, we know that  $\langle T'_{12} \xrightarrow{c'_2} T'_{22} \rangle^\bullet ((\lambda x : T'_{01}. e) (\langle T'_{11} \xrightarrow{c'_1} T'_{01} \rangle^\bullet e'_2))$  is equivalent to  $e_1 e'_1$ , so they both reduce to related results  $e'_1 \simeq_E e'_2 : T'_{12}$ . Now, by the IH on the codomain (along with E\_COERCE expansion and the observation that  $\text{coerce}(T'_{12}, T'_{22}, l) = c_2$ ), we know that  $\langle T_{12} \xrightarrow{\bullet} T_{22} \rangle^l e'_1 \simeq_E \langle T'_{12} \xrightarrow{c_2} T'_{22} \rangle^\bullet e'_2 : T'_{22}$ . We can apply coercion congruence again (Lemma B.23) to see that the behavior on  $e'_2$  is the same as the behavior on the unreduced term.  $\square$

### B.25 Lemma [Relating classic and eidetic source programs]:

1. If  $\Gamma \vdash_C e : T$  as a source program then  $\Gamma \vdash e \simeq_E e : T$ .
2. If  $\vdash_C T$  as a source program then  $T \sim_E T$ .

**Proof:** By mutual induction on the typing derivations.

#### Term typing $\boxed{\Gamma \vdash_C e : T}$

T\_VAR We have  $x = x$ . We know by assumption that  $\delta_1(x) \sim_E \delta_2(x) : T$ .

T\_CONST We have  $k = k$ . Since we are dealing with a source program,  $T = \{x:B \mid \text{true}\}$ . We have immediately that  $\text{ty}(k) = B$  and  $\text{true}[k/x] \simeq_E \text{true}[k/x] : \{x:\text{Bool} \mid \text{true}\}$  by reflexivity of  $\rightarrow_C^*$ , so  $k \simeq_E k : \{x:B \mid \text{true}\}$ .

T\_ABS Let  $\Gamma \models_E \delta$ . We must show that  $\lambda x : T_1. \delta_1(e_1) \sim_E \lambda x : T_1. \delta_2(e_1) : T_1 \rightarrow T_2$ . Let  $e_2 \sim_E e'_2 : T_1$ . We must show that applying the abstractions to these values yields related values. Both sides step by E\_BETA, to  $\delta_1(e_1)[e_2/x]$  and  $\delta_2(e_1)[e'_2/x]$ , respectively. But  $\Gamma, x : T_1 \models_E \delta[e_2, e'_2/x]$ , so we can apply IH (1) on  $e_1$  and  $e_1$  to show the two sides reduce to related results.

T\_OP By IH (1) on each argument, either one of the arguments goes to blame (in both calculi), and we are done by E\_OPRAISE, or all of the arguments reduce to related values. Since  $\text{ty}(op)$  is first order, these values must be related at refined base types, which means that they are in fact all equal constants. We then reduce by E\_OP on both sides to have  $\llbracket op \rrbracket(k_1, \dots, k_n)$ . We have assumed that the denotations of operations agree with their typings in *all* modes, so then  $\llbracket op \rrbracket(k_1, \dots, k_n)$  satisfies the refinement for  $\rightarrow_C$  in particular, and we are done.

T\_APP Let  $\Gamma \models_E \delta$ . We must show that  $\delta_1(e_1) \delta_1(e_2) \simeq_E \delta_2(e_1) \delta_2(e_2) : T_2$ . But by IH (1) on  $e_1$  and  $e_2$ , we are done directly.

T\_CAST Let  $\Gamma \models_E \delta$ . By IH (1) on  $e$ , we know that  $\delta_1(e) \simeq_E \delta_2(e) : T_1$ , either  $\delta_1(e) \rightarrow_C^* \uparrow l'$  and  $\delta_2(e) \rightarrow_E^* \uparrow l'$  (and we are done) or  $\delta_1(e')$  and  $\delta_2(e')$  reduce to values  $e_1 \sim_E e_2 : T_1$ . By Lemma B.24 (using IH (2) on the types), we know that  $\langle T_1 \xrightarrow{\bullet} T_2 \rangle^l e_1 \sim_E \langle T_1 \xrightarrow{\bullet} T_2 \rangle^l e_2 : T_2$ , so each side must reduce to a result  $e'_1 \sim_E e'_2 : T_2$ . We have cast congruence on the classic side straightforwardly, finding:

$$\langle T_1 \xrightarrow{\bullet} T_2 \rangle^l \delta_1(e') \rightarrow_C^* \langle T_1 \xrightarrow{\bullet} T_2 \rangle^l e_1 \rightarrow_C^* e'_1$$

On the heedful side, we can apply our derived cast congruence (Lemma B.23) to find that  $\langle T_1 \xrightarrow{\bullet} T_2 \rangle^l e_2 \rightarrow_E^* e'_2$  and  $\delta_2(e) \rightarrow_E^* e_2$  imply that  $\langle T_1 \xrightarrow{\bullet} T_2 \rangle^l \delta_2(e) \rightarrow_E^* e'_2$ .

T\_BLAZE Contradiction—doesn't appear in source programs. Though in fact it is in the relation, since  $\uparrow l \simeq_E \uparrow l : T$  for any  $l$  and  $T$ .

T\_CHECK Contradiction—doesn't appear in source programs.

Term type extraction

$$\boxed{\text{types}(e) : \mathcal{P}(T)}$$

$$\begin{aligned} \text{types}(x) &= \emptyset \\ \text{types}(k) &= \emptyset \\ \text{types}(\lambda x:T. e) &= \text{types}(T) \cup \text{types}(e) \\ \text{types}(\langle T_1 \xrightarrow{a} T_2 \rangle^l e) &= \text{types}(T_1) \cup \text{types}(T_2) \cup \text{types}(a) \cup \text{types}(e) \\ \text{types}(e_1 e_2) &= \text{types}(e_1) \cup \text{types}(e_2) \\ \text{types}(\text{op}(e_1, \dots, e_n)) &= \bigcup_{1 \leq i \leq n} \text{types}(e_i) \\ \text{types}(\langle \{x:B \mid e_1\}, e_2, k \rangle^l) &= \text{types}(\{x:B \mid e_1\}) \cup \text{types}(e_2) \\ \text{types}(\langle \{x:B \mid e_1\}, s, r, k, e \rangle^\bullet) &= \text{types}(\{x:B \mid e_1\}) \cup \text{types}(r) \cup \text{types}(e) \\ \text{types}(\uparrow l) &= \emptyset \end{aligned}$$

Type, type set, and coercion type extraction

$$\boxed{\text{types}(T) : \mathcal{P}(T)}$$

$$\begin{aligned} \text{types}(\{x:B \mid e\}) &= \{\{x:B \mid e\}\} \cup \text{types}(e) \\ \text{types}(T_1 \rightarrow T_2) &= \{T_1 \rightarrow T_2\} \cup \text{types}(T_1) \cup \text{types}(T_2) \end{aligned}$$

$$\boxed{\text{types}(a) : \mathcal{P}(T)}$$

$$\begin{aligned} \text{types}(\bullet) &= \emptyset \\ \text{types}(\mathcal{S}) &= \bigcup_{T \in \mathcal{S}} \text{types}(T) \\ \text{types}(\text{nil}) &= \emptyset \\ \text{types}(\{x:B \mid e\}^l, r) &= \{\{x:B \mid e\}\} \cup \text{types}(r) \\ \text{types}(c_1 \mapsto c_2) &= \text{types}(c_1) \cup \text{types}(c_2) \end{aligned}$$

Type height

$$\boxed{\text{height}(T)}$$

$$\begin{aligned} \text{height}(\{x:B \mid e\}) &= 1 \\ \text{height}(T_1 \rightarrow T_2) &= 1 + \max_{i \in \{1,2\}} \text{height}(T_i) \end{aligned}$$

Figure 14: Type extraction and type height

Type well formedness  $\boxed{\vdash_C T}$

WF\_BASE We can immediately see  $\text{true}[k/x] \simeq_E \text{true}[k/x] : \{x:\text{Bool} \mid \text{true}\}$  for any  $k \sim_E k : \{x:B \mid \text{true}\}$ , i.e., any  $k$  such that  $\text{ty}(k) = B$ , since  $\rightarrow_C^*$  is reflexive.

WF\_REFINE By inversion, we know that  $x:\{x:B \mid \text{true}\} \vdash_C e : \{x:\text{Bool} \mid \text{true}\}$ ; by IH (1), we find that  $\delta_1(e) \simeq_E \delta_2(e) : \{x:\text{Bool} \mid \text{true}\}$ , i.e., that  $e[e_1/x] \simeq_E e[e_2/x] : \{x:\text{Bool} \mid \text{true}\}$  for all  $e_1 \sim_E e_2 : \{x:B \mid \text{true}\}$ —which is what we needed to know.

WF\_FUN By IH (2) on each of the types.

□

## C Proofs of bounds for space-efficiency

This section contains our definitions for collecting types in a program and the corresponding proof of bounded space consumption (for all modes at once).

We define a function collecting all of the distinct types that appear in a program in Figure 14. If the type  $T = \{x:\text{Int} \mid x \geq 0\} \rightarrow \{y:\text{Int} \mid y \neq 0\}$  appears in the program  $e$ , then  $\text{types}(e)$  includes the type  $T$  itself along with its subparts  $\{x:\text{Int} \mid x \geq 0\}$  and  $\{y:\text{Int} \mid y \neq 0\}$ .

**C.1 Lemma:**  $\text{types}(e[e'/x]) \subseteq \text{types}(e) \cup \text{types}(e')$

**Proof:** By induction on  $e$ .

( $e = y$ )  $\text{types}(y) = \emptyset$ , so we must show that  $\text{types}(y[e'/x]) \subseteq \text{types}(e')$ . If  $x \neq y$ , then  $\text{types}(y[e'/x]) = \text{types}(y) = \emptyset$ , which is a subset of everything. If  $x = y$ , then  $\text{types}(x[e'/x]) = \text{types}(e')$ .

( $e = k$ ) Immediate, since  $k$  is closed and  $\text{types}(k) = \emptyset$ .

( $e = \lambda y:T. e$ ) By the IH on  $e$  and the closure of  $T$ .

( $e = \langle T_1 \xrightarrow{a} T_2 \rangle^l e$ ) By the IH on  $e$  and the closure of the types and annotations.

( $e = e_1 e_2$ ) By the IHs on  $e_1$  and  $e_2$ .

( $e = \text{op}(e_1, \dots, e_n)$ ) By the IHs on each  $e_i$ .

( $e = \langle \{x:B \mid e_1\}, e_2, k \rangle^l$ ) By the IH on  $e_2$  and the closure of  $\{x:B \mid e_1\}$ —noting that  $e_2$  is in fact closed in well typed terms.

( $e = \langle \{x:B \mid e_1\}, s, r, k, e \rangle^\bullet$ ) By the IH on  $e$ —though all of the terms are closed.

( $e = \uparrow l$ ) Immediate since  $\uparrow l$  is closed and  $\text{types}(\uparrow l) = \emptyset$ .

□

**C.2 Lemma:**  $\text{types}(T_1) \cup \text{types}(\text{merge}_m(T_1, a_1, T_2, a_2, T_3)) \cup \text{types}(T_3) \subseteq \text{types}(T_1) \cup \text{types}(a_1) \cup \text{types}(T_2) \cup \text{types}(a_2) \cup \text{types}(T_3)$

**Proof:** By cases on each, but observing that the merged annotation is always no bigger than the original, and that the type  $T_2$  may or may not vanish. □

**C.3 Lemma:**  $\text{types}(\text{dom}(a)) \subseteq \text{types}(a)$

**Proof:** This property is trivial when  $a = \bullet$ .

When the annotation is a type set, for  $\text{dom}(\mathcal{S})$  to be defined, every type in  $\mathcal{S}$  must be a function type. So:

$$\begin{aligned} \text{types}(\mathcal{S}) &= \bigcup_{T \in \mathcal{S}} \text{types}(T) \\ &= \bigcup_{T_1 \rightarrow T_2 \in \mathcal{S}} \text{types}(T_1 \rightarrow T_2) \\ &= \bigcup_{T_1 \rightarrow T_2 \in \mathcal{S}} \{T_1 \rightarrow T_2\} \cup \\ &\quad \text{types}(T_1) \cup \text{types}(T_2) \\ &\supseteq \bigcup_{T_1 \rightarrow T_2 \in \mathcal{S}} \text{types}(T_1) \\ &= \bigcup_{T \in \text{types}(\text{dom}(\mathcal{S}))} \text{types}(T) \\ &= \text{types}(\text{dom}(\mathcal{S})) \end{aligned}$$

Immediate when  $a = c_1 \mapsto c_2$ . □

**C.4 Lemma:**  $\text{types}(\text{cod}(a)) \subseteq \text{types}(a)$

**Proof:** This property is trivial when  $a = \bullet$ .

When the annotation is a type set, for  $\text{cod}(a)$  to be defined, every type in  $a$  must be a function type. So:

$$\begin{aligned} \text{types}(\mathcal{S}) &= \bigcup_{T \in \mathcal{S}} \text{types}(T) \\ &= \bigcup_{T_1 \rightarrow T_2 \in \mathcal{S}} \text{types}(T_1 \rightarrow T_2) \\ &= \bigcup_{T_1 \rightarrow T_2 \in \mathcal{S}} \{T_1 \rightarrow T_2\} \cup \text{types}(T_1) \cup \text{types}(T_2) \\ &\supseteq \bigcup_{T_1 \rightarrow T_2 \in \mathcal{S}} \text{types}(T_2) \\ &= \bigcup_{T \in \text{types}(\text{cod}(\mathcal{S}))} \text{types}(T) \\ &= \text{types}(\text{cod}(\mathcal{S})) \end{aligned}$$

Immediate when  $a = c_1 \mapsto c_2$ . □

**C.5 Lemma [Coercing types doesn't introduce types]:**

$\text{types}(\text{coerce}(T_1, T_2, l)) \subseteq \text{types}(T_1) \cup \text{types}(T_2)$

**Proof:** By induction on  $T_1$  and  $T_2$ . When they are refinements, we have the coercion just being  $\{x:B \mid e_2\}^l$ . When they are functions, by the IH. □

**C.6 Lemma [Dropping types doesn't introduce types]:**

$\text{types}(r \setminus \{x:B \mid e\}) \subseteq \text{types}(r)$

**Proof:** By induction on  $r$ .

( $r = \text{nil}$ ) The two sides are immediately equal.

( $r = \{x:B \mid e'\}^l, r'$ ) If  $\{x:B \mid e'\} \not\supseteq \{x:B \mid e\}$ , then the two are identical. If not, then we have  $\text{types}(r') \subseteq \text{types}(r)$  by the IH. □

**C.7 Lemma [Coercion merges don't introduce types]:**

$\text{types}(r_1 \triangleright r_2) \subseteq \text{types}(r_1) \cup \text{types}(r_2)$

**Proof:** By induction on  $r_1$ .

( $r_1 = \text{nil}$ ) The two sides are immediately equal.

( $r_1 = \{x:B \mid e\}^l, r'_1$ ) Using Lemma C.6, we find:

$$\begin{aligned} \text{types}(r_1 \triangleright r_2) &= \{\{x:B \mid e\}\} \cup \\ &\quad \text{types}(r'_1 \triangleright (r_2 \setminus \{x:B \mid e\})) \\ &\subseteq \{\{x:B \mid e\}\} \cup \text{types}(r'_1) \cup \\ &\quad \text{types}(r_2 \setminus \{x:B \mid e\}) \\ &\subseteq \{\{x:B \mid e\}\} \cup \text{types}(r'_1) \cup \text{types}(r_2) \\ &= \text{types}(r_1) \cup \text{types}(r_2) \end{aligned}$$

□

**C.8 Lemma [Reduction doesn't introduce types]:** If  $e \rightarrow_m e'$  then  $\text{types}(e') \subseteq \text{types}(e)$ .

**Proof:** By induction on the step taken.

**Shared rules**

(E\_BETA)

$$\begin{aligned} \text{types}((\lambda x:T. e_{12}) e_2) &= \text{types}(T) \cup \text{types}(e_{12}) \cup \text{types}(e_2) \\ &\supseteq \text{types}(e_{12}) \cup \text{types}(e_2) \quad (\text{Lemma C.1}) \\ &= \text{types}(e_{12}[e_2/x]) \end{aligned}$$

(E\_OP)

$$\begin{aligned} \text{types}(\text{op}(e_1, \dots, e_n)) &= \bigcup_{1 \leq i \leq n} \text{types}(e_i) \\ &\supseteq \emptyset \\ &= \text{types}(k) \quad (\text{operations are first order}) \\ &= \text{types}(\llbracket \text{op} \rrbracket(e_1, \dots, e_n)) \end{aligned}$$

(E\_UNWRAP)

$$\begin{aligned} &\text{types}(\langle T_{11} \rightarrow T_{12} \xRightarrow{a} T_{21} \rightarrow T_{22} \rangle^l e_1 e_2) \\ &= \text{types}(T_{11} \rightarrow T_{12}) \cup \text{types}(T_{21} \rightarrow T_{22}) \cup \text{types}(a) \cup \text{types}(e_1) \cup \text{types}(e_2) \\ &= \{T_{11} \rightarrow T_{12}\} \cup \text{types}(T_{11}) \cup \text{types}(T_{12}) \cup \{T_{21} \rightarrow T_{22}\} \cup \text{types}(T_{21}) \cup \text{types}(T_{22}) \cup \text{types}(a) \cup \text{types}(e_1) \cup \text{types}(e_2) \\ &\supseteq \text{types}(T_{11}) \cup \text{types}(T_{12}) \cup \text{types}(T_{21}) \cup \text{types}(T_{22}) \cup \text{types}(a) \cup \text{types}(e_1) \cup \text{types}(e_2) \\ &\supseteq \text{types}(T_{11}) \cup \text{types}(T_{12}) \cup \text{types}(T_{21}) \cup \text{types}(T_{22}) \cup \text{types}(\text{dom}(a)) \cup \text{types}(\text{cod}(a)) \cup \text{types}(e_1) \cup \text{types}(e_2) \\ &\quad (\text{Lemmas C.3 and C.4}) \\ &= \text{types}(\langle T_{12} \xRightarrow{\text{cod}(a)} T_{22} \rangle^l (e_1 (\langle T_{21} \xRightarrow{\text{dom}(a)} T_{11} \rangle^l e_2))) \end{aligned}$$

(E\_CHECKNONE)

$$\begin{aligned} \text{types}(\langle \{x:B \mid e_1\} \xRightarrow{\bullet} \{x:B \mid e_2\} \rangle^l k) &= \text{types}(\{x:B \mid e_1\}) \cup \text{types}(\bullet) \cup \text{types}(\{x:B \mid e_2\}) \cup \text{types}(k) \\ &\supseteq \text{types}(\{x:B \mid e_2\}) \cup \text{types}(k) \\ &= \{\{x:B \mid e_2\}\} \cup \text{types}(e_2) \cup \text{types}(k) \\ &= \text{types}(\{x:B \mid e_2\}) \cup \text{types}(e_2) \cup \text{types}(k) \\ &\supseteq \text{types}(\{x:B \mid e_2\}) \cup \text{types}(e_2[k/x]) \quad (\text{Lemma C.1}) \\ &= \text{types}(\{x:B \mid e_2\}) \quad \text{since } \text{types}(k) = \bullet \\ &= \text{types}(\langle \{x:B \mid e_2\}, e_2[k/x], k \rangle^l) \end{aligned}$$



(E\_CHECKOK)

$$\begin{aligned} \text{types}(\langle \{x:B \mid e\}, \text{true}, k \rangle^l) &= \text{types}(\{x:B \mid e\}) \cup \text{types}(\text{true}) \\ &\supseteq \emptyset \\ &= \text{types}(k) \end{aligned}$$

(E\_CHECKFAIL)

$$\begin{aligned} \text{types}(\langle \{x:B \mid e\}, \text{false}, k \rangle^l) &= \text{types}(\{x:B \mid e\}) \cup \text{types}(\text{false}) \\ &\supseteq \emptyset \\ &= \text{types}(\uparrow l) \end{aligned}$$

(E\_APPL)

$$\begin{aligned} \text{types}(e_1 \ e_2) &= \text{types}(e_1) \cup \text{types}(e_2) \\ &\supseteq \text{types}(e'_1) \cup \text{types}(e_2) \quad (\text{IH}) \\ &= \text{types}(e'_1 \ e_2) \end{aligned}$$

(E\_APPR)

$$\begin{aligned} \text{types}(e_1 \ e_2) &= \text{types}(e_1) \cup \text{types}(e_2) \\ &\supseteq \text{types}(e_1) \cup \text{types}(e'_2) \quad (\text{IH}) \\ &= \text{types}(e_1 \ e'_2) \end{aligned}$$

(E\_OPINNER)

$$\begin{aligned} \text{types}(\text{op}(e_1, \dots, e_{i-1}, e_i, \dots, e_n)) &= \bigcup_{1 \leq i \leq n} \text{types}(e_i) \\ &\supseteq \bigcup_{1 \leq j \leq i} \text{types}(e_j) \cup \text{types}(e'_i) \cup \bigcup_{i+1 \leq j \leq n} \text{types}(e_j) \quad (\text{IH}) \\ &= \text{types}(\text{op}(e_1, \dots, e_{i-1}, e'_i, \dots, e_n)) \end{aligned}$$

(E\_CHECKINNER)

$$\begin{aligned} \text{types}(\langle \{x:B \mid e_1\}, e_2, k \rangle^l) &= \text{types}(\{x:B \mid e_1\}) \cup \text{types}(e_2) \\ &\supseteq \text{types}(\{x:B \mid e_1\}) \cup \text{types}(e'_2) \quad (\text{IH}) \\ &= \text{types}(\langle \{x:B \mid e_1\}, e'_2, k \rangle^l) \end{aligned}$$

(E\_APPRAISEL)

$$\begin{aligned} \text{types}(\uparrow l \ e_2) &= \text{types}(\uparrow l) \cup \text{types}(e_2) \\ &\supseteq \emptyset \\ &= \text{types}(\uparrow l) \end{aligned}$$

(E\_APPRAISER)

$$\begin{aligned} \text{types}(e_1 \ \uparrow l) &= \text{types}(e_1) \cup \text{types}(\uparrow l) \\ &\supseteq \emptyset \\ &= \text{types}(\uparrow l) \end{aligned}$$

(E\_CASTRAISE)

$$\begin{aligned} \text{types}(\langle T_1 \xRightarrow{S} T_2 \rangle^l \ \uparrow l') &= \text{types}(T_1) \cup \text{types}(T_2) \cup \text{types}(S) \cup \text{types}(\uparrow l') \\ &\supseteq \emptyset \\ &= \text{types}(\uparrow l') \end{aligned}$$

(E\_OPRAISE)

$$\begin{aligned} \text{types}(\text{op}(e_1, \dots, e_{i-1}, \uparrow l, \dots, e_n)) &= \bigcup_{1 \leq j \leq i} \text{types}(e_j) \cup \text{types}(\uparrow l) \cup \bigcup_{i+1 \leq j \leq n} \text{types}(e_j) \\ &\supseteq \emptyset \\ &= \text{types}(\uparrow l) \end{aligned}$$

(E\_CHECKRAISE)

$$\begin{aligned} \text{types}(\langle \{x:B \mid e\}, \uparrow l, k \rangle^{l'}) &= \text{types}(\{x:B \mid e\}) \cup \text{types}(\uparrow l) \\ &\supseteq \emptyset \\ &= \text{types}(\uparrow l) \end{aligned}$$

### Classic $\lambda_H$ rules

(E\_CASTINNERC)

$$\begin{aligned} \text{types}(\langle T_1 \xrightarrow{\bullet} T_2 \rangle^l e) &= \text{types}(T_1) \cup \text{types}(T_2) \cup \text{types}(\bullet) \cup \text{types}(e) \\ &\supseteq \text{types}(T_1) \cup \text{types}(T_2) \cup \text{types}(\bullet) \cup \text{types}(e') \quad (\text{IH}) \\ &= \text{types}(\langle T_1 \xrightarrow{\bullet} T_2 \rangle^l e') \end{aligned}$$

### Efficient $\lambda_H$ rules ( $m \neq C$ )

(E\_CASTINNER)

$$\begin{aligned} \text{types}(\langle T_2 \xrightarrow{a} T_3 \rangle^l e_2) &= \text{types}(T_2) \cup \text{types}(T_3) \cup \text{types}(a) \cup \text{types}(e_2) \\ &\supseteq \text{types}(T_2) \cup \text{types}(T_3) \cup \text{types}(a) \cup \text{types}(e'_2) \quad (\text{IH}) \\ &= \text{types}(\langle T_2 \xrightarrow{a} T_3 \rangle^l e'_2) \end{aligned}$$

(E\_CASTMERGE)

$$\begin{aligned} \text{types}(\langle T_2 \xrightarrow{a_2} T_3 \rangle^l (\langle T_1 \xrightarrow{a_1} T_2 \rangle^{l'} e_2)) &= \text{types}(T_2) \cup \text{types}(T_3) \cup \text{types}(a_2) \cup \\ &\quad \text{types}(T_1) \cup \text{types}(T_2) \cup \text{types}(a_1) \cup \text{types}(e_2) \\ &\supseteq \text{types}(T_1) \cup \text{types}(T_3) \cup \\ &\quad \text{types}(\text{merge}_m(T_1, a_1, T_2, a_2, T_3)) \cup \text{types}(e_2) \quad (\text{Lemma C.2}) \\ &= \text{types}(\langle T_1 \xrightarrow{\text{merge}_m(T_1, a_1, T_2, a_2, T_3)} T_3 \rangle^l e_2) \end{aligned}$$

### Heedful $\lambda_H$ rules

(E\_TYPESET)

$$\begin{aligned} \text{types}(\langle T_1 \xrightarrow{\bullet} T_2 \rangle^l e) &= \text{types}(T_1) \cup \text{types}(T_2) \cup \text{types}(e) \\ &= \text{types}(\langle T_1 \xrightarrow{\emptyset} T_2 \rangle^l e) \end{aligned}$$

(E\_CHECKEMPTY)

$$\begin{aligned} \text{types}(\langle \{x:B \mid e_1\} \xrightarrow{\emptyset} \{x:B \mid e_2\} \rangle^l k) &= \text{types}(\{x:B \mid e_1\}) \cup \text{types}(\emptyset) \cup \text{types}(\{x:B \mid e_2\}) \cup \text{types}(k) \\ &\supseteq \text{types}(\{x:B \mid e_2\}) \cup \text{types}(k) \\ &= \{\{x:B \mid e_2\}\} \cup \text{types}(e_2) \cup \text{types}(k) \\ &= \text{types}(\{x:B \mid e_2\}) \cup \text{types}(e_2) \cup \text{types}(k) \\ &\supseteq \text{types}(\{x:B \mid e_2\}) \cup \text{types}(e_2[k/x]) \quad (\text{Lemma C.1}) \\ &= \text{types}(\{x:B \mid e_2\}) \quad \text{since } \text{types}(k) = \emptyset \\ &= \text{types}(\langle \{x:B \mid e_2\}, e_2[k/x], k \rangle^l) \end{aligned}$$

(E\_CHECKSET)

$$\begin{aligned} \text{types}(\langle \{x:B \mid e_1\} \xrightarrow{S} \{x:B \mid e_3\} \rangle^l k) &= \text{types}(\{x:B \mid e_1\}) \cup \text{types}(\{x:B \mid e_3\}) \cup \text{types}(S) \cup \text{types}(k) \\ &= \text{types}(\{x:B \mid e_1\}) \cup \text{types}(\{x:B \mid e_3\}) \cup \text{types}(S) \\ &= \text{types}(\{x:B \mid e_1\}) \cup \text{types}(\{x:B \mid e_3\}) \cup \\ &\quad \text{types}(S \setminus \{x:B \mid e_2\}) \cup \text{types}(\{x:B \mid e_2\}) \quad (\{x:B \mid e_2\} \in S) \\ &\supseteq \text{types}(\{x:B \mid e_1\}) \cup \text{types}(\{x:B \mid e_3\}) \cup \\ &\quad \text{types}(S \setminus \{x:B \mid e_2\}) \cup \text{types}(\{x:B \mid e_2\}) \cup \text{types}(e_2[k/x]) \\ &\quad (\text{Lemma C.1 and } \text{types}(k) = \emptyset) \\ &= \text{types}(\langle \{x:B \mid e_2\} \xrightarrow{S \setminus \{x:B \mid e_2\}} \{x:B \mid e_3\} \rangle^l \langle \{x:B \mid e_2\}, e_2[k/x], k \rangle^l) \end{aligned}$$

### Eidetic $\lambda_H$ rules

(E\_COERCE)

$$\begin{aligned} \text{types}(\langle T_1 \xrightarrow{\bullet} T_2 \rangle^l e) &= \text{types}(T_1) \cup \text{types}(T_2) \cup \text{types}(e) \\ &= \text{types}(\text{coerce}(T_1, T_2, l)) \cup \text{types}(T_1) \cup \text{types}(T_2) \cup \text{types}(e) \quad (\text{Lemma C.5}) \\ &= \text{types}(\langle T_1 \xrightarrow{\text{coerce}(T_1, T_2, l)} T_2 \rangle^{\bullet} e) \end{aligned}$$

(E\_COERCESTACK)

$$\begin{aligned}
\text{types}(\langle \{x:B \mid e_1\} \Rightarrow^r \{x:B \mid e_2\} \rangle^\bullet k) &= \text{types}(\{x:B \mid e_1\}) \cup \text{types}(\{x:B \mid e_2\}) \cup \text{types}(r) \\
&\supseteq \text{types}(\{x:B \mid e_2\}) \cup \text{types}(r) \\
&= \langle \{x:B \mid e_2\}, ?, r, k, k \rangle^\bullet
\end{aligned}$$

(E\_STACKDONE)

$$\begin{aligned}
\text{types}(\langle \{x:B \mid e\}, \checkmark, \text{nil}, k, k \rangle^\bullet) &= \text{types}(\{x:B \mid e\}) \\
&\supseteq \emptyset \\
&= \text{types}(k)
\end{aligned}$$

(E\_STACKPOP)

$$\begin{aligned}
\text{types}(\langle \{x:B \mid e\}, s, (\{x:B \mid e'\}^l, r), k, k \rangle^\bullet) &= \text{types}(\{x:B \mid e\}) \cup \text{types}(\{x:B \mid e'\}^l, r) \\
&= \text{types}(\{x:B \mid e\}) \cup \text{types}(\{x:B \mid e'\}) \cup \text{types}(r) \cup \text{types}(e') \\
&= \text{types}(\{x:B \mid e\}) \cup \text{types}(\{x:B \mid e'\}) \cup \text{types}(r) \cup \text{types}(e'[k/x]) \quad (\text{Lemma C.1}) \\
&= \text{types}(\langle \{x:B \mid e\}, s \vee (e = e'), r, k, \langle \{x:B \mid e'\}, e'[k/x], k \rangle^l \rangle^\bullet)
\end{aligned}$$

(E\_STACKINNER)

$$\begin{aligned}
\text{types}(\langle \{x:B \mid e\}, s, r, k, e' \rangle^\bullet) &= \text{types}(\{x:B \mid e\}) \cup \text{types}(r) \cup \text{types}(e') \\
&\supseteq \text{types}(\{x:B \mid e\}) \cup \text{types}(r) \cup \text{types}(e'') \quad (\text{IH}) \\
&= \text{types}(\langle \{x:B \mid e\}, s, r, k, e'' \rangle^\bullet)
\end{aligned}$$

(E\_STACKRAISE)

$$\begin{aligned}
\text{types}(\langle \{x:B \mid e\}, \checkmark, \text{nil}, k, k \rangle^\bullet) &= \text{types}(\{x:B \mid e\}) \\
&\supseteq \emptyset \\
&= \text{types}(k)
\end{aligned}$$

□